

EQUIVARIANT COMPLEX STRUCTURES ON HOMOGENEOUS SPACES AND THEIR COBORDISM CLASSES

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ABSTRACT. We consider compact homogeneous spaces G/H , where G is a compact connected Lie group and H is its closed connected subgroup of maximal rank. The aim of this paper is to provide an effective computation of the universal toric genus for the complex, almost complex and stable complex structures which are invariant under the canonical left action of the maximal torus T^k on G/H . As it is known, on G/H we may have many such structures and the computations of their toric genus in terms of fixed points for the same torus action give the constraints on possible collections of weights for the corresponding representations of T^k in the tangent spaces at the fixed points, as well as on the signs at these points. In that context, the effectiveness is also approached due to an explicit description of the relations between the weights and signs for an arbitrary couple of such structures. Special attention is devoted to the structures which are invariant under the canonical action of the group G . Using classical results, we obtain an explicit description of the weights and signs in this case. We consequently obtain an expression for the cobordism classes of such structures in terms of coefficients of the formal group law in cobordisms, as well as in terms of Chern numbers in cohomology. These computations require no information on the cohomology ring of the manifold G/H , but, on their own, give important relations in this ring. As an application we provide an explicit formula for the cobordism classes and characteristic numbers of the flag manifolds $U(n)/T^n$, Grassmann manifolds $G_{n,k} = U(n)/(U(k) \times U(n-k))$ and some particular interesting examples.

1. INTRODUCTION

The homogeneous spaces G/H , where G is a compact connected Lie group and H is its closed connected subgroup of maximal rank are classical manifolds, which play significant role in many areas of mathematics. They can be characterized as homogeneous spaces with positive Euler characteristic. Our interest in these manifolds is related to the well known problem in cobordism theory to find representatives in cobordism classes that have reach geometric structure and carry many non-equivalent stable complex structures. Let us mention [2] that all compact homogeneous spaces G/H where H is centralizer of an element of odd order admit an invariant almost complex structure. Furthermore, if H is a centralizer of a maximal torus in G then G/H admits an invariant complex structure which gives rise to a unique invariant Kaehler-Einstein metric and moreover all homogeneous complex spaces are algebraic. Besides that by [31], the stationary subgroup H of any homogeneous complex space G/H can be realized as the fixed point set under the action of some finite group of inner automorphisms of G and vice versa. This shows that these spaces can be made off generalized symmetric spaces what results in the existence of finite order symmetries on them. Our interest in the study of the homogeneous spaces G/H with positive Euler characteristic is also stimulated by the well known relations between the cohomology rings of these spaces and the deep problems in the theory of representations and combinatorics (see, for example [14]). These problems are formulated in terms of different additive basis in the cohomology rings for

G/H and multiplicative rules related to that basis. We hope that the relations in the cohomology rings $H^*(G/H, \mathbb{Z})$ obtained from the calculation of the universal toric genus of the manifold G/H may lead to new results in that direction.

In the paper [28], which opened a new stage in the development of the cobordism theory, S. P. Novikov proposed a method for the description of the fixed points for actions of groups on manifolds, based on the formal group law for geometric cobordisms. That paper rapidly stimulated active research work which brought significant results. These results in the case of S^1 -actions are mainly contained in the papers [15], [16], [21], [22] and also in [23]. Our approach to this problem uses the results on the universal toric genus, which was introduced in [5] and described in details in [7]. Let us note that the formula for the universal toric genus in terms of fixed points is a generalization of Krichever's formula [21] to the case of stable complex manifolds. For the description of the cobordism classes of manifolds in terms of their Chern numbers we appeal to the Chern-Dold character theory, which is developed in [4].

The universal toric genus can be constructed for any even dimensional manifold M^{2n} with a given torus action and stable complex structure which is equivariant under the torus action. Moreover, if the set of isolated fixed points for this action is finite than the universal toric genus can be localized, which means that it can be explicitly written through the weights and the signs at the fixed points for the representations that gives arise from the given torus action.

The construction of the toric genus is reduced to the computation of Gysin homomorphism of 1 in complex cobordisms for fibration whose fiber is M^{2n} and the base is classifying space of the torus. The problem of the localization of Gysin homomorphism is very known and it was studied by many authors, starting with 60-es of the last century. In [5] and [7] is obtained explicit answer for this problem in the terms of the torus action on tangent bundle for M^{2n} . The history of this problem is presented also in these papers.

If consider compact homogeneous space G/H with $\text{rk } G = \text{rk } H = k$, we have on it the canonical action θ of the maximal torus T^k for H and G , and any G -invariant almost complex structure on G/H is compatible with this action. Besides that, all fixed points for the action θ are isolated, so one can apply localization formula to compute universal toric genus for this action and any invariant almost complex structure. Since, in this case, we consider almost complex structures, all fixed points in the localization formula are going to have sign $+1$. We prove that the weights for the action θ at different fixed points can be obtained by the action of the Weyl group W_G up to the action of the Weyl group W_H on the weights for θ at identity fixed point. On this way we get an explicit formula for the cobordism classes of such spaces in terms of the weights at the fixed point eH . This formula also shows that the cobordism class for G/H related to an invariant almost complex structure can be computed without information about cohomology for G/H .

We obtain also the explicit formulas, in terms of the weights at identity fixed point, for the cohomology characteristic numbers for homogeneous spaces of positive Euler characteristic endowed with an invariant almost complex structure. We use further that the cohomology characteristic numbers s_ω , $\omega = (i_1, \dots, i_n)$, and classical Chern numbers $c^\omega = c_1^{i_1} \dots c_n^{i_n}$ are related by some standard relations from the theory of symmetric polynomials. This fact together with the obtained formulas for the characteristic numbers $s_\omega(\tau(G/H))$ proves that the classical Chern numbers $c^\omega(\tau(G/H))$ for the almost complex homogeneous spaces can be computed without information on their cohomology. It also gives an explicit way for the computation of the classical Chern numbers.

In studying invariant almost complex structures on compact homogeneous spaces G/H of positive Euler characteristic we appeal to the theory developed in [8] which describes such structures in terms of complementary roots for G related to H . In that context by an invariant almost complex structure on G/H we assume the structure that is invariant under the canonical action of G on G/H .

In this paper we go further with an application of the universal toric genus and consider the almost complex structures on G/H for which we require to be invariant only under the canonical action θ of the maximal torus T on G/H , as well as the stable complex structures equivariant under this torus action. We prove generally that for an arbitrary M^{2n} the weights for any two such structures differ only by the signs. We also prove that the sign difference in the weights determines, up to common factor ± 1 , the difference of the signs at the fixed points related to any two such structures.

We provide an application of our results by obtaining explicit formula for the cobordism class and top cohomology characteristic number of the flag manifolds $U(n)/T^n$ and Grassmann manifolds $G_{n,k} = U(n)/(U(k) \times U(n-k))$ related to the standard complex structures. We want to emphasize that, our method when applying to the flag manifolds and Grassmann manifolds gives the description of their cobordism classes and characteristic numbers using the technique of *divided difference operators*. Our method also makes possible to compare cobordism classes that correspond to the different invariant almost complex structures on the same homogeneous space. We illustrate that on the space $U(4)/(U(1) \times U(1) \times U(2))$, which is firstly given in [8] as an example of homogeneous space that admits two different invariant complex structures.

We also compute the universal toric genus of the sphere $S^6 = G_2/SU(3)$ related to unique G_2 -invariant almost complex structure for which is known not to be complex [8]. We prove more, that this structure is also unique almost complex structure invariant under the canonical action θ of the maximal torus T^2 .

This paper comes out from the first part of our work where we mainly considered invariant almost complex structures on homogeneous spaces of positive Euler characteristic. It has continuation which is going to deal with the same questions, but related to the stable complex structures equivariant under given torus action on homogeneous spaces of positive Euler characteristic.

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2. UNIVERSAL TORIC GENUS

We will recall the results from [5], [6] and [7].

2.1. General setting. In general setting one considers $2n$ -dimensional manifold M^{2n} with a given smooth action θ of the torus T^k . We say that (M^{2n}, θ, c_τ) is *equivariant stable complex* if it admits θ -equivariant stable complex structure c_τ . This means that there exists $l \in \mathbb{N}$ and complex vector bundle ξ such that

$$(1) \quad c_\tau: \tau(M^{2n}) \oplus \mathbb{R}^{2(l-n)} \longrightarrow \xi$$

is real isomorphism and the composition

$$(2) \quad r(t): \xi \xrightarrow{c_\tau^{-1}} \tau(M^{2n}) \oplus \mathbb{R}^{2(l-n)} \xrightarrow{d\theta(t) \oplus I} \tau(M^{2n}) \oplus \mathbb{R}^{2(l-n)} \xrightarrow{c_\tau} \xi$$

is a complex transformation for any $t \in T^k$.

If there exists ξ such that $c_\tau: \tau(M^{2n}) \longrightarrow \xi$ is an isomorphism, i. e. $l = n$, then (M^{2n}, θ, c_τ) is called *almost complex* T^k -manifold.

Denote by $\Omega_U^*[[u_1, \dots, u_k]]$ an algebra of formal power series over $\Omega_U^* = U^*(pt)$. It is well known [27] that $U^*(pt) = \Omega_U^* = \mathbb{Z}[y_1, \dots, y_n, \dots]$, where $\dim y_n = -2n$. Moreover, as the generators for Ω_U^* over the rationals, or in other words for $\Omega_U^* \otimes \mathbb{Q}$, can be taken the family of cobordism classes $[CP^n]$ of the complex projective spaces.

When given a θ -equivariant stable complex structure c_τ on M^{2n} , we can always choose θ -equivariant embedding $i: M^{2n} \rightarrow \mathbb{R}^{2(n+m)}$, where $m > n$, such that c_τ determines, up to natural equivalence, a θ -equivariant complex structure c_ν on the normal bundle $\nu(i)$ of i . Therefore, one can define the universal toric genus for (M^{2n}, θ, c_τ) in complex cobordisms, see [5], [7].

We want to note that, in the case when c_τ is almost complex structure, a universal toric genus for (M^{2n}, θ, c_τ) is completely defined in terms of the action θ on tangent bundle $\tau(M^{2n})$.

The universal toric genus for (M^{2n}, θ, c_τ) could be looked at as an element in algebra $\Omega_U^*[[u_1, \dots, u_k]]$. It is defined with

$$(3) \quad \Phi(M^{2n}, \theta, c_\tau) = [M^{2n}] + \sum_{|\omega| > 0} [G_\omega(M^{2n})] u^\omega,$$

where $\omega = (i_1, \dots, i_k)$ and $u^\omega = u_1^{i_1} \dots u_k^{i_k}$.

Here by $[M^{2n}]$ is denoted the complex cobordism class of the manifold M^{2n} with stable complex structure c_τ , by $G_\omega(M^{2n})$ is denoted the stable complex manifold obtained as the total space of the fibration $G_\omega \rightarrow B_\omega$ with fiber M . The base $B_\omega = \prod_{j=1}^k B_j^{i_j}$, where $B_j^{i_j}$ is Bott tower, i. e. i_j -fold iterated two-sphere bundle over $B_0 = pt$. The base B_ω satisfies $[B_\omega] = 0$, $|\omega| > 0$, where $|\omega| = \sum_{j=1}^k i_j$.

For the universal toric genus of homogeneous space of positive Euler characteristic we prove the following.

Lemma 1. *Let $M^{2n} = G/H$, where G is compact connected Lie group and H its closed connected subgroup of maximal rank. Denote by θ the canonical action of the maximal torus T^k on M^{2n} and let c_τ be G -equivariant stable complex structure on M^{2n} . Then the universal toric genus $\Phi(M^{2n}, \theta, c_\tau)$ belongs to the image of homomorphism $Bj^*: U^*(BG) \rightarrow U^*(BT^k)$ which is induced by the embedding $T^k \subset G$.*

Proof. According to its construction, the universal toric genus $\Phi(M^{2n}, \theta, c_\tau)$ is equal to $p_!(1)$ and belongs to $U^{-2n}(BT^k)$, where

$$p: ET^k \times_{T^k} M^{2n} \rightarrow BT^k.$$

Using that the action θ is induced by the left action of the group G and looking at the commutative diagram

$$\begin{array}{ccc} ET^k & \rightarrow & EG \\ \downarrow & & \downarrow \\ BT^k & \rightarrow & BG \end{array}$$

we obtain the proof due to the fact that Gysin homomorphism is functorial for the bundles that can be connected with commutative diagram. \square

2.2. The action with isolated fixed points. We first introduce, following [5], the general notion of the *sign* at isolated fixed point. Let we are on M^{2n} given an equivariant stable complex structure c_τ .

We assume $\mathbb{R}^{2(l-n)}$ in (1) to be endowed with canonical orientation. Under this assumption the real isomorphism (1) defines an orientation on $\tau(M^{2n})$, since ξ is canonically oriented by an existing complex structure.

On the other hand, if p is the isolated fixed point, the representation $r_p: T^k \rightarrow GL(l, \mathbb{C})$ associated to (2) produces the decomposition of the fiber $\xi_p \cong \mathbb{C}^l$ as $\xi_p \cong \mathbb{C}^{l-n} \oplus \mathbb{C}^n$. In this decomposition r_p acts trivially on \mathbb{C}^{l-n} and without trivial summands on \mathbb{C}^n . Note that \mathbb{C}^n inherits here the complex structure from ξ_p which defines an orientation of \mathbb{C}^n . This together leads to the following definition.

Definition 1. The $\text{sign}(p)$ at isolated fixed point p is $+1$ if the map

$$\tau_p(M^{2n}) \xrightarrow{I \oplus 0} \tau_p(M^{2n}) \oplus \mathbb{R}^{2(l-n)} \xrightarrow{c_{\tau,p}} \xi_p \cong \mathbb{C}^n \oplus \mathbb{C}^{l-n} \xrightarrow{\pi} \mathbb{C}^n ,$$

preserves orientation. Otherwise, $\text{sign}(p)$ is -1 .

Remark 1. Note that for an almost complex T^k -manifold M^{2n} , it directly follows from the definition that $\text{sign}(p) = +1$ for any isolated fixed point.

If an action θ of T^k on M^{2n} has only isolated fixed points, then it is proved that toric genus for M^{2n} can be completely described using just local data at the fixed points, [5], [7].

Namely, let p again be an isolated fixed point. Then the non trivial summand of r_p from (2) gives rise to the tangential representation of T^k in $GL(n, \mathbb{C})$. This representation decomposes into n non-trivial one-dimensional representations of $r_{p,1} \oplus \dots \oplus r_{p,n}$ of T^k . Each of the representations $r_{p,j}$ can be written as

$$r_{p,j}(e^{2\pi i x_1}, \dots, e^{2\pi i x_k})v = e^{2\pi i \langle \Lambda_j(p), \mathbf{x} \rangle} v ,$$

for some $\Lambda_j(p) = (\Lambda_j^1(p), \dots, \Lambda_j^k(p)) \in \mathbb{Z}^k$, where $\mathbf{x} = (x_1, \dots, x_k) \in \mathbb{R}^k$ and $\langle \Lambda_j(p), \mathbf{x} \rangle = \sum_{l=1}^k \Lambda_j^l(p) x_l$. The sequence $\{\Lambda_1(p), \dots, \Lambda_n(p)\}$ is called *the weight vector* for representation r_p in the fixed point p .

Remark 2. Since p is an isolated fixed point none of the couples of weights $\Lambda_i(p)$, $1 \leq i \leq n$, have common integer factor.

Theorem 1. Let c_τ^1 and c_τ^2 be the stable complex structures on M^{2n} equivariant under the given action θ of the torus T^k on the manifold M^{2n} with isolated fixed points.

- The weights $\Lambda_i^1(p)$ and $\Lambda_i^2(p)$, $1 \leq i \leq n$, of an action θ at the fixed point p corresponding to the structures c_τ^1 and c_τ^2 are related with

$$(4) \quad \Lambda_i^2(p) = a_i(p) \Lambda_i^1(p), \text{ where } a_i(p) = \pm 1 .$$

- The signs $\text{sign}(p)_1$ and $\text{sign}(p)_2$ at the fixed point p corresponding to the structures c_τ^1 and c_τ^2 are related with

$$(5) \quad \text{sign}(p)_2 = \epsilon \cdot \prod_{i=1}^n a_i(p) \cdot \text{sign}(p)_1 ,$$

where $\epsilon = \pm 1$ depending if c_τ^1 and c_τ^2 define of $\tau(M^{2n})$ the same orientation or not. Here $a_i(p)$ are such that $\Lambda_i^2(p) = a_i(p)\Lambda_i^1(p)$, for the weights $\Lambda_i^1(p)$ and $\Lambda_i^2(p)$ of an action θ related to the structures c_τ^1 and c_τ^2 .

Proof. Let p be an isolated fixed point of an action θ of the torus T^k on the manifold M^{2n} . If it is on M^{2n} given the θ -equivariant stable complex structure c_{τ_1} than in the neighborhood of p , the tangential representation of T^k in $GL(n, \mathbb{C})$ assigned to an action θ and structure c_{τ_1} decomposes into the sum of non-trivial one-dimensional representations $r_{p,1} \oplus \dots \oplus r_{p,n}$. Any other stable complex structure c_{τ_2} which is equivariant under the given action θ commutes with each of the one-dimensional representations $r_{p,i}$, $1 \leq i \leq n$. Therefore, the one-dimensional summands in which decomposes the tangential representation of T^k , assigned to an action θ and structure c_{τ_2} , are $r_{p,i}$ or it's conjugate $\overline{r_{p,i}}$, $1 \leq i \leq n$. This implies that the relations between the weights for an action θ related to the two different stable θ -equivariant stable complex structures are given by the formula (4).

To prove the second statement of the Theorem let us note that the sign at the fixed point p of some θ -equivariant stable complex structure c_τ is determined by the orientations of the real two-dimensional subspaces in which decomposes summand \mathbb{C}^n of $\xi_p = \mathbb{C}^n \oplus \mathbb{C}^{l-n}$. That decomposition is obtained using the decomposition of the tangential representation of T^k determined by the action θ and the structure c_τ . Therefore, by (4) it follows that the relation between the signs at the fixed point for the given torus action related to the two equivariant stable complex structures is given by (5). \square

Remark 3. We want to point that Theorem 1 gives that, under assumption that manifold M^{2n} admits θ -equivariant stable complex structure, the signs at the fixed points for any other θ -equivariant stable complex structure are completely determined by an orientation that structure defines on M^{2n} and by the weights at fixed points. In other words, when passing from the existing to some other θ -equivariant stable complex structure, the "correction" of the sign at arbitrary fixed point is completely determined by the "correction" of the weights at that fixed point up to some common factor $\epsilon = \pm 1$ which points on the difference of orientations on M^{2n} that these two structures define.

2.3. Formal group law. Let $F(u, v) = u + v + \sum \alpha_{ij} u^i v^j$ be the formal group for complex cobordism [28]. The corresponding power system $\{[w](u) \in \Omega^*[[u]] : w \in \mathbb{Z}\}$ is uniquely defined with $[0](u) = 0$ and $[w](u) = F(u, [w-1](u))$, for $w \in \mathbb{Z}$. For $\mathbf{w} = (w_1, \dots, w_k) \in \mathbb{Z}^k$ and $\mathbf{u} = (u_1, \dots, u_k)$ one defines $[\mathbf{w}](\mathbf{u})$ inductively with $[\mathbf{w}](\mathbf{u}) = [w](u)$ for $k = 1$ and

$$[\mathbf{w}](\mathbf{u}) = F_{q=1}^k [w_q](u_q) = F(F_{q=1}^{k-1} [w_q](u_q), [w_k](u_k)) ,$$

for $k \geq 2$. Then for toric genus of the action θ with isolated fixed points the following localization formula holds, which is first formulated in [5] and proved in details in [7].

Theorem 2. *If the action θ has a finite set P of isolated fixed points then*

$$(6) \quad \Phi(M^{2n}, \theta, c_\tau) = \sum_{p \in P} \text{sign}(p) \prod_{j=1}^n \frac{1}{[\Lambda_j(p)](\mathbf{u})} .$$

Remark 4. Theorem 2 together with formula (3) gives that

$$(7) \quad \sum_{p \in P} \text{sign}(p) \prod_{j=1}^n \frac{1}{[\Lambda_j(p)](\mathbf{u})} = [M^{2n}] + \mathfrak{L}(\mathbf{u}) ,$$

where $\mathfrak{L}(\mathbf{u}) \in \Omega_U^*[[u_1, \dots, u_k]]$ and $\mathfrak{L}(\mathbf{0}) = 0$. In this way Theorem 2 gives that all summands in the left hand side of (7) have order n in 0.

Remark 5. As we will make it explicit further, the fact that after making the sum, all singularities in formula (6) should disappear, gives constraints on the weights and signs at the fixed points. Note also, that formula (6) gives an expression for the cobordism class $[M^{2n}]$ in terms of the weights and signs at fixed points.

2.4. Chern-Dold character. We show further how one, together with Theorem 2, can use the notion of Chern-Dold character in cobordisms in order to obtain an expression for cobordism class $[M^{2n}]$ in terms of the characteristic numbers for M^{2n} , as well as the relations on the weights and signs at fixed points. In review of the basic definitions and results on Chern character we follow [4].

Let U^* be the theory of unitary cobordisms.

Definition 2. The Chern-Dold character for a topological space X in the theory of unitary cobordisms U^* is a ring homomorphism

$$(8) \quad ch_U : U^*(X) \rightarrow H^*(X, \Omega_U^* \otimes \mathbb{Q}) .$$

Recall that the Chern-Dold character as a multiplicative transformation of cohomology theories is uniquely defined by the condition that for $X = (pt)$ it gives canonical inclusion $\Omega_U^* \rightarrow \Omega_U^* \otimes \mathbb{Q}$.

The Chern-Dold character splits into composition

$$(9) \quad ch_U : U^*(X) \rightarrow H^*(X, \Omega_U^*(\mathbb{Z})) \rightarrow H^*(X, \Omega_U^* \otimes \mathbb{Q}) .$$

The ring $\Omega_U^*(\mathbb{Z})$ in (9) is firstly described in [4]. It is a subring of $\Omega_U^* \otimes \mathbb{Q}$ generated by the elements from $\Omega_U^{-2n} \otimes \mathbb{Q}$ having integer Chern numbers. It is equal to

$$\Omega_U^*(\mathbb{Z}) = \mathbb{Z}[b_1, \dots, b_n, \dots] ,$$

where $b_n = \frac{1}{n+1}[\mathbb{C}P^n]$.

The Chern character leaves $[M^{2n}]$ invariant, i. e.

$$ch_U([M^{2n}]) = [M^{2n}] ,$$

and ch_U is the homomorphism of Ω_U^* -modules

It follows from the its description [4] that the Chern-Dold character $ch_U : U^*(X) \rightarrow H^*(X, \Omega_U^*(\mathbb{Z}))$ as a multiplicative transformation of the cohomology theories is given by the series

$$ch_U u = h(x) = \frac{x}{f(x)}, \quad \text{where} \quad f(x) = 1 + \sum_{i=1}^{\infty} a_i x^i \quad \text{and} \quad a_i \in \Omega_U^{-2i}(\mathbb{Z}) .$$

Here $u = c_1^U(\eta) \in U^2(\mathbb{C}P^\infty)$ and $x = c_1^H(\eta) \in H^2(\mathbb{C}P^\infty, \mathbb{Z})$ denote the first Chern classes of the universal complex line bundle $\eta \rightarrow \mathbb{C}P^\infty$.

From the construction of Chern-Dold character it follows also the equality

$$(10) \quad ch_U[M^{2n}] = [M^{2n}] = \sum_{\|\omega\|=n} s_\omega(\tau(M^{2n})) a^\omega ,$$

where $\omega = (i_1, \dots, i_n)$, $\|\omega\| = \sum_{l=1}^n l \cdot i_l$ and $a^\omega = a_1^{i_1} \cdots a_n^{i_n}$. Here the numbers $s_\omega(\tau(M^{2n}))$, $\|\omega\| = n$ are the cohomology characteristic numbers of M^{2n} and they correspond to the cohomology tangent characteristic classes of M^{2n} .

If on M^{2n} is given torus action θ of T^k and stable complex structure c_τ which is θ -equivariant, then the Chern character of its toric genus is

$$(11) \quad ch_U \Phi(M^{2n}, \theta, c_\tau) = [M^{2n}] + \sum_{|\xi| > 0} [G_\xi(M^{2n})] (ch_U \mathbf{u})^\xi,$$

where $ch_U \mathbf{u} = (ch_U u_1, \dots, ch_U u_k)$, $ch_U u_i = \frac{x_i}{f(x_i)}$ and $\xi = (i_1, \dots, i_k)$, $|\xi| = i_1 + \dots + i_k$.

We have that $F(u, v) = g^{-1}(g(u) + g(v))$, where $g(u) = u + \sum_{n > 0} \frac{1}{n+1} [\mathbb{C}P^n] u^{n+1}$ (see [28]) is the logarithm of the formal group $F(u, v)$ and $g^{-1}(u)$ is the exponent of $F(u, v)$, that is the function inverse to the series $g(u)$. Using that $ch_U g(u) = g(ch_U(u)) = g(\frac{x}{f(x)}) = x$ (see [4]), we obtain $g^{-1}(x) = \frac{x}{f(x)}$ and $ch_U F(u_1, u_2) = \frac{x_1 + x_2}{f(x_1 + x_2)}$ and therefore

$$ch_U [\Lambda_j(p)](u) = \frac{\langle \Lambda_j(p), \mathbf{x} \rangle}{f(\langle \Lambda_j(p), \mathbf{x} \rangle)}.$$

Applying these results to Theorem 2 we get

$$(12) \quad ch_U \Phi(M^{2n}, \theta, c_\tau) = \sum_{p \in P} \text{sign}(p) \prod_{j=1}^n \frac{f(\langle \Lambda_j(p), \mathbf{x} \rangle)}{\langle \Lambda_j(p), \mathbf{x} \rangle}.$$

From (11) and (12) it follows that

$$(13) \quad \sum_{p \in P} \text{sign}(p) \prod_{j=1}^n \frac{f(\langle \Lambda_j(p), \mathbf{x} \rangle)}{\langle \Lambda_j(p), \mathbf{x} \rangle} = [M^{2n}] + \sum_{|\xi| > 0} [G_\xi(M^{2n})] (ch_U \mathbf{u})^\xi.$$

Example 1. Let us take $M^2 = \mathbb{C}P^1 = U(2)/(U(1) \times U(1))$. We have the action θ of T^2 on $\mathbb{C}P^1$ with two fixed points. The weights related to the standard complex structure c_τ , are $(x_1 - x_2)$ and $(x_2 - x_1)$. By equation (12) we obtain that the Chern character of the universal toric genus for $(\mathbb{C}P^1, \theta, c_\tau)$ is given by the series

$$ch_U \Phi(\mathbb{C}P^1, \theta, c_\tau) = \frac{f(x_1 - x_2)}{x_1 - x_2} + \frac{f(x_2 - x_1)}{x_2 - x_1} = 2 \sum_{k=0}^{\infty} a_{2k+1} (x_1 - x_2)^{2k}.$$

By equation (11) we obtain

$$[\mathbb{C}P^1] + \sum_{i+j > 0} [G_{i,j}(\mathbb{C}P^1)] \frac{x_1^i x_2^j}{f(x_1)^i f(x_2)^j} = 2a_1 + 2 \sum_{k=1}^{\infty} a_{2k+1} (x_1 - x_2)^{2k}.$$

Thus, $[\mathbb{C}P^1] = 2a_1$ and $\sum_{i+j=n} [G_{i,j}(\mathbb{C}P^1)] = 0$ for any $n > 0$. Moreover,

$$\begin{aligned} [G_{i,j}(\mathbb{C}P^1)] &\sim 0, \text{ if } i + j = 2k + 1, \\ [G_{i,2k-j}(\mathbb{C}P^1)] &\sim (-1)^i 2 \binom{2k}{i} a_{2k+1}, \quad k > 0, \end{aligned}$$

where “ \sim ” is equality to the elements decomposable in $\Omega_U(\mathbb{Z})$.

Note that the subgroup $S^1 = \{(t_1, t_2) \in T^2 | t_2 = 1\}$ acts also on $\mathbb{C}P^1$ with the same fixed points as for the action of T^2 , but the weights are going to be x and $-x$. This action is, as an example, given in [5].

If in the left hand side of the equation (13) we put tx instead of x and then multiply it with t^n we obtain the following result.

Proposition 1. *The coefficient for t^n in the series in t*

$$\sum_{p \in P} \text{sign}(p) \prod_{j=1}^n \frac{f(t\langle \Lambda_j(p), \mathbf{x} \rangle)}{\langle \Lambda_j(p), \mathbf{x} \rangle}$$

represents the complex cobordism class $[M^{2n}]$.

Proposition 2. *The coefficient for t^l in the series in t*

$$\sum_{p \in P} \text{sign}(p) \prod_{j=1}^n \frac{f(t\langle \Lambda_j(p), \mathbf{x} \rangle)}{\langle \Lambda_j(p), \mathbf{x} \rangle}$$

is equal to zero for $0 \leq l \leq n - 1$.

3. TORUS ACTION ON HOMOGENEOUS SPACES WITH POSITIVE EULER CHARACTERISTIC.

Let G/H be a compact homogeneous space of positive Euler characteristic. It means that G is a compact connected Lie group and H its connected closed subgroup, such that $\text{rk } G = \text{rk } H$. Let T be the maximal common torus for G and H . There is canonical action θ of T on G/H given by $t(gH) = (tg)H$, where $t \in T$ and $gH \in G/H$. Denote by $N_G(T)$ the normalizer of the torus T in G . Then $W_G = N_G(T)/T$ is the Weyl group for G . For the set of fixed points for the action θ we prove the following.

Proposition 3. *The set of fixed points under the canonical action θ of T on G/H is given by $(N_G(T)) \cdot H$.*

Proof. It is easy to see that gH is fixed point for θ for any $g \in N_G(T)$. If gH is the fixed point under the canonical action of T on G/H then $t(gH) = gH$ for all $t \in T$. It follows that $g^{-1}tg \in H$ for all $t \in T$, i. e. $g^{-1}Tg \subset H$. This gives that $g^{-1}Tg$ is a maximal torus in H and, since any two maximal toruses in H are conjugate, it follows that $g^{-1}Tg = h^{-1}Th$ for some $h \in H$. Thus, $(gh)^{-1}T(gh) = T$ what means that $gh \in N_G(T)$. But, $(gh)H = gH$, what proves the statement. \square

Since $T \subset N_G(T)$ leaves H fixed, the following Lemma is direct implication of the Proposition 3.

Lemma 2. *The set of fixed points under the canonical action θ of T on G/H is given by $W_G \cdot H$.*

Regarding the number of fixed points, it holds the following.

Lemma 3. *The number of fixed points under the canonical action θ of T on G/H is equal to the Euler characteristic $\chi(G/H)$.*

Proof. Let $g, g' \in N_G(T)$ are representatives of the same fixed point. Then $g'g^{-1} \in H$ and $g^{-1}Tg = T = (g')^{-1}Tg'$, what gives that $g'g^{-1}Tg(g')^{-1} = T$ and, thus, $g'g^{-1} \in N_H(T)$. This implies that the number of fixed points is equal to

$$\left\| \frac{N_G(T)}{N_H(T)} \right\| = \frac{\left\| \frac{N_G(T)}{T} \right\|}{\left\| \frac{N_H(T)}{T} \right\|} = \frac{\|W_G\|}{\|W_H\|} = \chi(G/H) .$$

The last equality is classical result related to equal ranks homogeneous spaces, see [30]. \square

Remark 6. The proof of Lemma 3 gives that the set of fixed points under the canonical action θ of T on G/H can be obtained as an orbit of eH by the action of the Weyl group W_G up to the action of the Weyl group W_H .

4. THE WEIGHTS AT THE FIXED POINTS.

Denote by \mathfrak{g} , \mathfrak{h} and \mathfrak{t} the Lie algebras for G , H respectively and $T = T^k$, where $k = \text{rk } G = \text{rk } H$. Let $\alpha_1, \dots, \alpha_m$ be the roots for \mathfrak{g} related to \mathfrak{t} , where $\dim G = 2m + k$. Recall that the roots for \mathfrak{g} related to \mathfrak{t} are the weights for the adjoint representation Ad_T of T which is given with $\text{Ad}_T(t) = d_e \text{ad}(t)$, where $\text{ad}(t)$ are inner automorphisms of G defined by the elements $t \in T$. One can always choose the roots for G such that $\alpha_{n+1}, \dots, \alpha_m$ gives the roots for \mathfrak{h} related to \mathfrak{t} , where $\dim H = 2(m - n) + k$. The roots $\alpha_1, \dots, \alpha_n$ are called the *complementary* roots for \mathfrak{g} related to \mathfrak{h} . Using root decomposition for \mathfrak{g} and \mathfrak{h} it follows that $T_e(G/H) \cong \mathfrak{g}_{\alpha_1}^{\mathbb{C}} \oplus \dots \oplus \mathfrak{g}_{\alpha_n}^{\mathbb{C}}$, where by \mathfrak{g}_{α_i} is denoted the root subspace defined with the root α_i and $T_e(G/H)$ is the tangent space for G/H at the $e \cdot H$. It is obvious that $\dim_{\mathbb{R}} G/H = 2n$.

4.1. Description of the invariant almost complex structures. Assume we are given an invariant almost complex structure J on G/H . This means that J is invariant under the canonical action of G on G/H . Then according to the paper [8], we can say the following.

- Since J is invariant it commutes with adjoint representation Ad_T of the torus T . This implies that J induces the complex structure on each complementary root subspace $\mathfrak{g}_{\alpha_1}, \dots, \mathfrak{g}_{\alpha_n}$. Therefore, J can be completely described by the root system $\varepsilon_1 \alpha_1, \dots, \varepsilon_n \alpha_n$, where we take $\varepsilon_i = \pm 1$ depending if J and adjoint representation Ad_T define the same orientation on \mathfrak{g}_{α_i} or not, $1 \leq i \leq n$. The roots $\varepsilon_k \alpha_k$ are called *the roots of the almost complex structure J* .
- If we assume J to be integrable, it follows that it can be chosen the ordering on the canonical coordinates of \mathfrak{t} such that the roots $\varepsilon_1 \alpha_1, \dots, \varepsilon_n \alpha_n$ which define J make the closed system of positive roots.

Let us assume that G/H admits an invariant almost complex structure. Consider the isotropy representation I_e of H in $T_e(G/H)$ and let it decomposes into s *real irreducible representations* $I_e = I_e^1 + \dots + I_e^s$. Then it is proved in [8] that G/H admits exactly 2^s invariant almost complex structures. Because of completeness we recall the proof of this fact shortly here. Consider the decomposition of $T_e(G/H)$

$$T_e(G/H) = \mathfrak{J}_1 \oplus \dots \oplus \mathfrak{J}_s$$

such that the restriction of I_e on \mathfrak{J}_i is I_e^i . The subspaces $\mathfrak{J}_1, \dots, \mathfrak{J}_s$ are invariant under T and therefore each of them is the sum of some root subspaces, i.e. $\mathfrak{J}_i = \mathfrak{g}_{\alpha_{i_1}} \oplus \dots \oplus \mathfrak{g}_{\alpha_{i_j}}$, for some complementary roots $\alpha_{i_1}, \dots, \alpha_{i_j}$. Any linear transformation that commutes with I_e leaves each of \mathfrak{J}_i invariant. Since, by assumption G/H admits invariant almost complex structure, we have at least one linear transformation without real eigenvalue that commutes with I_e . This implies that the commuting field for each of I_e^i is the field of complex numbers and, thus, on each \mathfrak{J}_i we have exactly two invariant complex structures.

Remark 7. Note that this consideration shows that the numbers $\varepsilon_1, \dots, \varepsilon_n$ that define an invariant almost complex structure may not vary independently.

Remark 8. In this paper we consider almost complex structures on G/H that are invariant under the canonical action of the group G , what, as we remarked, imposes some relations on $\varepsilon_1, \dots, \varepsilon_n$.

If we do not require G -invariance, but just T -invariance, we will have more degrees of freedom on $\varepsilon_1, \dots, \varepsilon_n$. This paper is going to have continuation, where, among the other, the case of T -invariant structures will be studied.

Example 2. Since the isotropy representation for $\mathbb{C}P^n$ is irreducible over the reals, it follows that on $\mathbb{C}P^n$ we have only two invariant almost complex structures, which are actually the standard complex structure and its conjugate.

Example 3. The flag manifold $U(n)/T^n$ admits 2^m invariant almost complex structures, where $m = \frac{n(n-1)}{2}$. By [8] only two of them, conjugate to each other, are integrable.

Example 4. As we already mentioned, the 10-dimensional manifold $M^{10} = U(4)/(U(1) \times U(1) \times U(2))$ is the first example of homogeneous space, where we have an existence of two non-equivalent invariant complex structures, see [8]. We will, in the last section of this paper, also describe cobordism class of M^{10} for these structures.

4.2. The weights at the fixed points. We fix now an invariant almost complex structure J on G/H and we want to describe the weights of the canonical action θ of T on G/H at the fixed points of this action. If gH is the fixed point for the action θ , then we have a linear map $d_g\theta(t) : T_g(G/H) \rightarrow T_g(G/H)$ for all $t \in T$. Therefore, this action gives rise to the complex representation $d_g\theta$ of T in $(T_g(G/H), J)$.

The weights for this representation at identity fixed point are described in [8].

Lemma 4. *The weights for the representation $d_e\theta$ of T in $(T_e(G/H), J)$ are given by the roots of an invariant almost complex structure J .*

Proof. Let us, because of clearness, recall the proof. The inner automorphism $\text{ad}(t)$, for $t \in T$ induces the map $\overline{\text{ad}}(t) : G/H \rightarrow G/H$ given with $\overline{\text{ad}}(t)(gH) = t(gH)t^{-1} = (tg)H$. Therefore, $\theta(t) = \overline{\text{ad}}(t)$ and, thus, $d_e\theta(t) = d_e\overline{\text{ad}}(t)$ for any $t \in T$. This directly gives that the weights for $d_e\theta$ in $(T_e(G/H), J)$ are the roots that define J . \square

For an arbitrary fixed point we prove the following.

Theorem 3. *Let gH be the fixed point for the canonical action θ of T on G/H . The weights of the induced representation $d_g\theta$ of T in $(T_g(G/H), J)$ can be obtained from the weights of the representation $d_e\theta$ of T in $(T_e(G/H), J)$ by the action of the Weyl group W_G up to the action of the Weyl group W_H .*

Proof. Note that Lemma 2 gives that an arbitrary fixed point can be written as wH for some $w \in W_G/W_H$. Fix $w \in W_G/W_H$ and denote by $l(w)$ the action of w on G/H , given by $l(w)gH = (wg)H$ and by $\text{ad}(w)$ the inner automorphism of G given by w .

We observe that $\theta \circ \text{ad}(w) = \text{ad}(w) \circ \theta$ and therefore $d_e\theta \circ d_e\text{ad}(w) = d_e\text{ad}(w) \circ d_e\theta$. This implies that the weights for $d_e\theta \circ d_e\text{ad}(w)$ we get by the action of $d_e\text{ad}(w)$ on the weights for $d_e\theta$. From the other side $\theta(\text{ad}(w)t)gH = (w^{-1}twg)H = (l(w^{-1}) \circ \theta(t) \circ l(w))gH$ what implies that $d_e(\theta \circ \text{ad}(w)) = d_w l(w^{-1}) \circ d_w\theta \circ d_e l(w)$. This gives that if, using the map $d_w l_{w^{-1}}$, we lift the weights for $d_w\theta$ from $T_w(G/H)$ to $T_e(G/H)$, we get that they coincide with the weights for $d_e\theta \circ d_e\text{ad}(w)$. Therefore, the weights for $d_w\theta$ we can get by the action of the element w on the weights for $d_e\theta$. \square

5. THE COBORDISM CLASSES OF HOMOGENEOUS SPACES WITH POSITIVE EULER CHARACTERISTIC

Theorem 4. *Let G/H be a homogeneous space of compact connected Lie group such that $\text{rk } G = \text{rk } H = k$, $\dim G/H = 2n$ and consider the canonical action θ of maximal torus $T = T^k$ for G and H on G/H . Assume we are given an invariant almost complex structure J on G/H . Let $\Lambda_j = \varepsilon_j \alpha_j$, $1 \leq j \leq n$, where $\varepsilon_1 \alpha_1, \dots, \varepsilon_n \alpha_n$ are the complementary roots of G related to H which define an invariant almost complex structure J . Then the toric genus for $(G/H, J)$ is given with*

$$(14) \quad \Phi(G/H, J) = \sum_{\mathbf{w} \in W_G/W_H} \prod_{j=1}^n \frac{1}{[\mathbf{w}(\Lambda_j)](\mathbf{u})}.$$

Proof. Rewriting Theorem 2, since all fixed points have sign $+1$, we get that the toric genus for $(G/H, J)$ is

$$(15) \quad \Phi(G/H, J) = \sum_{p \in P} \prod_{j=1}^n \frac{1}{[\Lambda_j(p)](\mathbf{u})},$$

where P is the set of isolated fixed points and $\{\Lambda_1(p), \dots, \Lambda_n(p)\}$ is the weight vector of the representation for T in $T_p(G/H)$ associated to an action θ . By Theorem 3, the set of fixed points P coincides with the orbit of the action of W_G/W_H on eH and also by Theorem 3 the set of weight vectors at fixed points coincides with the orbit of the action of W_G/W_H on the weight vector Λ at eH . The result follows if we put this data into formula (15). \square

Corollary 1. *The Chern-Dold character of the toric genus for homogeneous space $(G/H, J)$ is given with*

$$(16) \quad ch_U \Phi(G/H, J) = \sum_{\mathbf{w} \in W_G/W_H} \prod_{j=1}^n \frac{f(\langle \mathbf{w}(\Lambda_j), \mathbf{x} \rangle)}{\langle \mathbf{w}(\Lambda_j), \mathbf{x} \rangle},$$

where $f(t) = 1 + \sum_{i \geq 1} a_i t^i$ for $a_i \in \Omega_U^{-2i}(\mathbb{Z})$, $\mathbf{x} = (x_1, \dots, x_k)$ and by $\langle \Lambda_j, \mathbf{x} \rangle = \sum_{l=1}^k \Lambda_j^l x_l$ is denoted the weight vector Λ_j of T^k -representation at $e \cdot H$.

Corollary 2. *The cobordism class for $(G/H, J)$ is given as the coefficient for t^n in the series in t*

$$(17) \quad \sum_{\mathbf{w} \in W_G/W_H} \prod_{j=1}^n \frac{f(t \langle \mathbf{w}(\Lambda_j), \mathbf{x} \rangle)}{\langle \mathbf{w}(\Lambda_j), \mathbf{x} \rangle}.$$

Remark 9. Since the weights of different invariant almost complex structures on the fixed homogeneous space G/H differ only by sign, Corollary 2 provides the way for comparing cobordism classes of two such structures on G/H without having their cobordism classes explicitly computed.

6. CHARACTERISTIC NUMBERS OF HOMOGENEOUS SPACES WITH POSITIVE EULER CHARACTERISTIC.

6.1. Generally about stable complex manifolds. Let M^{2n} be an equivariant stable complex manifold whose given action θ of the torus T^k on M^{2n} has only isolated fixed points. Denote by P the

set of fixed points for θ and set $t_j(p) = \langle \Lambda_j(p), \mathbf{x} \rangle$, where $\{\Lambda_j(p), j = 1, \dots, n\}$ are the weight vectors of the representation of T^k at a fixed point p given by the action θ and $\mathbf{x} = (x_1, \dots, x_k)$.

Set

$$(18) \quad \prod_{i=1}^n f(t_i) = 1 + \sum f_\omega(t_1, \dots, t_n) a^\omega .$$

Using this notation Proposition 2 could be formulated in the following way.

Proposition 4. *For any ω with $0 \leq \|\omega\| \leq (n-1)$ we have that*

$$\sum_{p \in P} \text{sign}(p) \cdot \frac{f_\omega(t_1(p), \dots, t_n(p))}{t_1(p) \cdots t_n(p)} = 0 .$$

Note that Proposition 4 gives the strong constraints on the set of signs $\{\text{sign}(p)\}$ and the set of weights $\{\Lambda_j(p)\}$ at fixed points for a manifold with a given torus action and related equivariant stable complex structure. For example for $\omega = (i_1, \dots, i_n)$ such that $i_k = 1$ for exactly one k such that $1 \leq k \leq n-1$ and $i_j = 0$ for $j \neq k$ it gives that the signs and the weights at fixed points have to satisfy the following relations.

Corollary 3.

$$\sum_{p \in P} \text{sign}(p) \cdot \frac{\sum_{i=1}^n t_j^k(p)}{t_1(p) \cdots t_n(p)} = 0 ,$$

where $0 \leq k \leq n-1$.

As we already mentioned in (10) the cobordism class for M^{2n} can be represented as

$$[M^{2n}] = \sum_{\|\omega\|=n} s_\omega(\tau(M^{2n})) a^\omega ,$$

where $\omega = (i_1, \dots, i_n)$, $\|\omega\| = \sum_{l=1}^n l \cdot i_l$ and $a^\omega = a_1^{i_1} \cdots a_n^{i_n}$.

If the given action θ of T^k on M^{2n} is with isolated fixed points, the coefficients $s_\omega(\tau(M^{2n}))$ can be explicitly described using Proposition 1 and expression (18).

Theorem 5. *Let M^{2n} be an equivariant stable complex manifold whose given action θ of the T^k has only isolated fixed points. Denote by P the set of fixed points for θ and set $t_j(p) = \langle \Lambda_j(p), \mathbf{x} \rangle$, where $\Lambda_j(p)$ are the weight vectors of the representation of T^k at fixed points given by the action θ and $\mathbf{x} = (x_1, \dots, x_k)$. Then for $\|\omega\| = n$*

$$(19) \quad s_\omega(\tau(M^{2n})) = \sum_{p \in P} \text{sign}(p) \cdot \frac{f_\omega(t_1(p), \dots, t_n(p))}{t_1(p) \cdots t_n(p)} .$$

Example 5.

$$s_{(n,0,\dots,0)}(\tau(M^{2n})) = \sum_{p \in P} \text{sign}(p) .$$

Example 6.

$$s_{(0,\dots,0,1)}(\tau(M^{2n})) = s_n(M^{2n}) = \sum_{p \in P} \text{sign}(p) \frac{\sum_{j=1}^n t_j^n(p)}{t_1(p) \cdots t_n(p)} .$$

Remark 10. Note that the left hand side of (19) in Theorem 5 is an integer number $s_\omega(\tau(M^{2n}))$ while the right hand side is a rational function in variables x_1, \dots, x_k . So this theorem imposes strong restrictions on the sets of signs $\{\text{sign}(p)\}$ and weight vectors $\{\Lambda_j(p)\}$ at the fixed points.

6.1.1. On existence of more stable complex structures. Let assume that manifold M^{2n} endowed with torus action θ with isolated fixed points admits θ -equivariant stable complex structure c_τ . Denote by $\Lambda_1(p), \dots, \Lambda_n(p)$ the weights for an action θ at fixed points $p \in P$ and by $\text{sign}(p)$ the signs at the fixed points related to c_τ . Let further $t_j(p) = \langle \Lambda_j(p), \mathbf{x} \rangle$, $1 \leq j \leq n$. Then Theorem 1, Proposition 4 and Theorem 5 give the following necessary condition for the existence of another θ -equivariant stable complex structure on M^{2n} .

Proposition 5. *If M^{2n} admits an other θ -equivariant stable complex structure $(M^{2n}, c'_\tau, \theta)$ then there exist $a_i(p) = \pm 1$, where $p \in P$ and $1 \leq i \leq n$, such that the following conditions are satisfied:*

- for any ω with $0 \leq \|\omega\| \leq (n-1)$

$$(20) \quad \sum_{p \in P} \text{sign}(p) \frac{f_\omega(a_1(p)t_1(p), \dots, a_n(p)t_n(p))}{t_1(p) \cdots t_n(p)} = 0.$$

- for any $\|\omega\| = n$

$$(21) \quad \sum_{p \in P} \text{sign}(p) \frac{f_\omega(a_1(p)t_1(p), \dots, a_n(p)t_n(p))}{t_1(p) \cdots t_n(p)}$$

is an integer number.

As a special case we get analogue of Corollary 3.

Corollary 4. *If M^{2n} admits an other θ -equivariant stable complex structure $(M^{2n}, c'_\tau, \theta)$ then there exist $a_j(p) = \pm 1$, where $p \in P$ and $1 \leq j \leq n$, such that*

$$(22) \quad \sum_{p \in P} \text{sign}(p) \frac{\sum_{i=1}^n (a_j(p))^k t_j^k(p)}{t_1(p) \cdots t_n(p)} = 0,$$

for $1 \leq k \leq n-1$.

Remark 11. Note that the relations (22) only for k being odd give the constraints on the existence of the second stable complex structure. In the same spirit it follows from Example 6 that if n is even than the characteristic number $s_n(M^{2n})$ are the same for all stable complex structures on M^{2n} equivariant under the fixed torus action θ . For n being odd, as Example 13 and Subsection 8.3 will show, these numbers may be different.

6.2. Homogeneous spaces of positive Euler characteristic and with invariant almost complex structure. Let us assume M^{2n} to be homogeneous space G/H of positive Euler characteristic with canonical action of a maximal torus and endowed with an invariant almost complex structure J . All fixed points have sign $+1$ and taking into account Theorem 3, Proposition 4 gives that the weights at the fixed points have to satisfy the following relations.

Corollary 5. For any ω with $0 \leq \|\omega\| \leq (n-1)$ where $2n = \dim G/H$ we have that

$$(23) \quad \sum_{w \in W_G/W_H} w \left(\frac{f_\omega(t_1, \dots, t_n)}{t_1 \cdots t_n} \right) = 0 ,$$

where $t_j = \langle \Lambda_j, \mathbf{x} \rangle$ and Λ_j , $1 \leq j \leq n$, are the weights at the fixed point $e \cdot H$.

In the same way, Theorem 5 implies that

Theorem 6. For $M^{2n} = G/H$ and $t_j = \langle \Lambda_j, \mathbf{x} \rangle$, where $\langle \Lambda_j, \mathbf{x} \rangle = \sum_{l=1}^k \Lambda_j^l x_l$, $\mathbf{x} = (x_1, \dots, x_k)$, $k = \text{rk } G = \text{rk } H$, we have

$$(24) \quad s_\omega(\tau(M^{2n})) = \sum_{w \in W_G/W_H} w \left(\frac{f_\omega(t_1, \dots, t_n)}{t_1 \cdots t_n} \right)$$

for any ω such that $\|\omega\| = n$.

Example 7.

$$s_{(n,0,\dots,0)}(G/H, J) = \|W_G/W_H\| = \chi(G/H)$$

and, therefore, $s_{(n,0,\dots,0)}(G/H, J)$ does not depend on invariant almost complex structure J .

Corollary 6.

$$s_{(0,\dots,0,1)}(G/H, J) = s_n(G/H, J) = \sum_{w \in W_G/W_H} w \left(\frac{\sum_{j=1}^n t_j^n}{t_1 \cdots t_n} \right) .$$

Example 8. In the case $\mathbb{C}P^n = G/H$ where $G = U(n+1)$, $H = U(1) \times U(n)$ we have action of T^{n+1} and related to the standard complex structure the weights are given with $\langle \Lambda_j, \mathbf{x} \rangle = x_j - x_{n+1}$, $j = 1, \dots, n$ and $W_G/W_H = \mathbb{Z}_{n+1}$ is cyclic group. So

$$(25) \quad s_n(\mathbb{C}P^n) = \sum_{i=1}^{n+1} \frac{\sum_{j \neq i} (x_i - x_j)^n}{\prod_{j \neq i} (x_i - x_j)} = n + 1 .$$

Example 9. Let us consider Grassmann manifold $G_{q+2,2} = G/H$ where $G = U(q+2)$, $H = U(q) \times U(2)$. We have here the canonical action of the torus T^{q+2} . The weights for this action at identity point related to the standard complex structure are given with $\langle \Lambda_{ij}, \mathbf{x} \rangle = x_i - x_j$, where $1 \leq i \leq q$, $j = q+1, q+2$. There are $\|W_{U(q+2)}/W_{U(2) \times U(q)}\| = \frac{(q+2)(q+1)}{2}$ fixed points for this action. Therefore

$$(26) \quad s_{2q}(G_{q+2,2}) = \sum_{w \in W_{U(q+2)}/W_{U(2) \times U(q)}} w \left(\frac{\sum_{i=1}^q ((x_i - x_{q+1})^{2q} + (x_i - x_{q+2})^{2q})}{\prod_{i=1}^q (x_i - x_{q+1})(x_i - x_{q+2})} \right) .$$

The action of the group $W_{U(q+2)}/W_{U(2) \times U(q)}$ on the weights at the identity point in formula (26) is given by the permutations between the coordinates x_1, \dots, x_q and coordinates x_{q+1}, x_{q+2} . The explicit description of non trivial such permutations is as follows:

$$\begin{aligned} w_{k,q+1}(k) &= q+1, \quad w_{k,q+1}(q+1) = k, \quad \text{where } 1 \leq k \leq q, \\ w_{k,q+2}(k) &= q+2, \quad w_{k,q+2}(q+2) = k, \quad \text{where } 1 \leq k \leq q, \end{aligned}$$

$w_{k,l}(q+1) = k$, $w_{k,l}(k) = q+1$, $w_{k,l}(q+2) = l$, $w_{k,l}(l) = q+2$ for $1 \leq k \leq q-1$, $k+1 \leq l \leq q$.

As we remarked before (see Remark 10), the expression on the right hand side in (26) is an integer number, so we can get a value for s_{2q} by choosing the appropriate values for the vector (x_1, \dots, x_{q+2}) . For example, if we take $q = 2$ and $(x_1, x_2, x_3, x_4) = (1, 2, 3, 4)$ the straightforward application of formula (26) will give that $s_4(G_{4,2}) = -20$.

Example 10. In the case $G_{q+l,l} = G/H$, where $G = U(q+l)$, $H = U(q) \times U(l)$ we have

$$(27) \quad s_{lq}(G_{q+l,l}) = \sum_{\sigma \in S_{q+l}/(S_q \times S_l)} \sigma \left(\frac{\sum (x_i - x_j)^{lq}}{\prod (x_i - x_j)} \right),$$

where $1 \leq i \leq q$, $(q+1) \leq j \leq (q+l)$ and S_{q+l} is the symmetric group.

We consider later, in the Section 8, the case of this Grassmann manifold in more details.

6.2.1. Chern numbers. We want to deduce an explicit relations between cohomology characteristic numbers s_ω and classical Chern numbers for an invariant almost complex structure on G/H .

Proposition 6. *The number $s_\omega(\tau(M^{2n}))$, where $\omega = (i_1, \dots, i_n)$, $\|\omega\| = n$, is the characteristic number that corresponds to the characteristic class given by the orbit of the monomial*

$$(u_1 \cdots u_{i_1})(u_{i_1+1}^2 \cdots u_{i_1+i_2}^2) \cdots (u_{i_1+\dots+i_{n-1}+1}^n \cdots u_{i_1+\dots+i_n}^n).$$

Remark 12. Let $\xi = (j_1, \dots, j_n)$ and $\mathbf{u}^\xi = u_1^{j_1} \cdots u_n^{j_n}$. The orbit of the monomial \mathbf{u}^ξ is defined with

$$O(\mathbf{u}^\xi) = \sum \mathbf{u}^{\xi'},$$

where the sum is over the orbit $\{\xi' = \sigma\xi, \sigma \in S_n\}$ of the vector $\xi \in \mathbb{Z}^n$ under the symmetric group S_n acting by permutations of coordinates of ξ .

Example 11. If we take $\omega = (n, 0, \dots, 0)$ we need to compute the coefficient for a_1^n and it is given as an orbit $O(u_1 \cdots u_n)$ what is the elementary symmetric function σ_n . If we take $\omega = (0, \dots, 0, 1)$ then we should compute the coefficient for a_n and it is given with $O(u_1^n) = \sum_{j=1}^n u_j^n$, what is Newton polynomial.

It is well known fact from the algebra of symmetric functions that the orbits of monomials give the additive basis for the algebra of symmetric functions. Therefore, any orbit of monomial can be expressed through elementary symmetric functions and vice versa. It gives the expressions for the characteristic numbers s_ω in terms of Chern characteristic numbers $c^\omega = c_1^{i_1} \cdots c_n^{i_n}$ for an almost complex homogeneous space $(G/H, J)$.

Theorem 7. *Let $\omega = (i_1, \dots, i_n)$, $\|\omega\| = n$, and assume that the orbit of the monomial*

$$(u_1 \cdots u_{i_1})(u_{i_1+1}^2 \cdots u_{i_1+i_2}^2) \cdots (u_{i_1+\dots+i_{n-1}+1}^n \cdots u_{i_1+\dots+i_n}^n)$$

is expressed through the elementary symmetric functions as

$$(28) \quad O((u_1 \cdots u_{i_1})(u_{i_1+1}^2 \cdots u_{i_1+i_2}^2) \cdots (u_{i_1+\dots+i_{n-1}+1}^n \cdots u_{i_1+\dots+i_n}^n)) = \\ = \sum_{\|\xi\|=n} \beta_{\omega\xi} \sigma_1^{l_1} \cdots \sigma_n^{l_n}$$

for some $\beta_{\omega\xi} \in \mathbb{Z}$ and $\|\xi\| = \sum_{j=1}^n j \cdot l_j$, where $\xi = (l_1, \dots, l_n)$. Then it holds

$$(29) \quad s_{\omega}(G/H, J) = \sum_{w \in W_G/W_H} w\left(\frac{f_{\omega}(t_1, \dots, t_n)}{t_1 \dots t_n}\right) = \sum_{\|\xi\|=n} \beta_{\omega\xi} c_1^{l_1} \dots c_n^{l_n},$$

where c_i are the Chern classes for the tangent bundle of $(G/H, J)$.

Remark 13. Let $p(n)$ denote the number of partitions of the number n . By varying ω , the equation (29) gives the system of $p(n)$ linear equations in Chern numbers whose determinant is, by (28), non-zero. Therefore, it provides the explicit formulas for the computation of Chern numbers.

Remark 14. We want to point that relation (29) in Theorem 7 together with Theorem 6 proves that the Chern numbers for $(G/H, J)$ can be computed without having any information on cohomology for G/H .

Example 12. We provide the direct application of Theorem 7 following Example 11. It is straightforward to see that $s_{(n,0,\dots,0)}(G/H) = c_n(G/H)$ for any invariant almost complex structure. This together with Example 7 gives that $c_n(G/H) = \chi(G/H)$.

We want to add that it is given in [26] a description of the numbers s_I that correspond to our characteristic numbers s_{ω} , but the numerations I and ω are different. To the partition $i \in I$ correspond the n -tuple $\omega = (i_1, \dots, i_n)$ such that i_k is equal to the number of appearances of the number k in the partition i .

7. ON EQUIVARIANT STABLE COMPLEX HOMOGENEOUS SPACES OF POSITIVE EULER CHARACTERISTIC.

Assume we are given on G/H a stable complex structure c_{τ} which is equivariant under the canonical action θ of the maximal torus T . If $p = gH$ is the fixed point for an action θ , by Section 2 and Section 3, we see that $T_{gH}(G/H) = \mathfrak{g}_{w(\alpha_1)}^{\mathbb{C}} \oplus \dots \oplus \mathfrak{g}_{w(\alpha_n)}^{\mathbb{C}}$, where $w \in W_G/W_H$ and $\alpha_1, \dots, \alpha_n$ are the complementary roots for G related to H . The following statement is the direct consequence of Theorem 1 and Lemma 2.

Corollary 7. *Let $\alpha_1, \dots, \alpha_n$ be the set of complementary roots for G related to H . The set of weights of an action θ at the fixed points related to an arbitrary equivariant stable complex structure c_{τ} is of the form*

$$(30) \quad \{a_1(w) \cdot w(\alpha_1), \dots, a_n(w) \cdot w(\alpha_n)\},$$

where $w \in W_G/W_H$ and $a_i(w) = \pm 1$ for $1 \leq i \leq n$.

For the signs at the fixed points (which we identify with $w \in W_G/W_H$) using Theorem 1 we obtain the following.

Corollary 8. *Assume that G/H admits an invariant almost complex structure J defined by the complementary roots $\alpha_1, \dots, \alpha_n$. Let c_{τ} be the θ -equivariant stable complex structure with the set of weights $\{a_1(w) \cdot w(\alpha_1), \dots, a_n(w) \cdot w(\alpha_n)\}$, $w \in W_G/W_H$ at the fixed points. The signs at the fixed points are given with*

$$(31) \quad \text{sign}(w) = \epsilon \cdot \prod_{i=1}^n a_i(w), \quad w \in W_G/W_H,$$

where $\epsilon = \pm 1$ depending if J and c_{τ} define the same orientation on M^{2n} or not.

This implies the following consequence of Proposition 5.

Corollary 9. *Assume that G/H admits an invariant almost complex structure J defined by the complementary roots $\alpha_1, \dots, \alpha_n$. Let c_τ be the θ -equivariant stable complex structure with the set of weights $\{a_1(w) \cdot w(\alpha_1), \dots, a_n(w) \cdot w(\alpha_n)\}$, $w \in W_G/W_H$ at the fixed points. Then*

- *for any ω with $0 \leq \|\omega\| \leq (n-1)$ we have that*

$$(32) \quad \sum_{w \in W_G/W_H} \frac{f_\omega(a_1(w)w(\alpha_1), \dots, a_n(w)w(\alpha_n))}{w(\alpha_1) \cdots w(\alpha_n)} = 0 ;$$

- *for any $\|\omega\| = n$*

$$(33) \quad \sum_{w \in W_G/W_H} \frac{f_\omega(a_1(w)w(\alpha_1), \dots, a_n(w)w(\alpha_n))}{w(\alpha_1) \cdots w(\alpha_n)}$$

is an integer number.

Remark 15. Corollary 9 gives the strong constraints on the numbers $a_i(w)$ that appear as the "coefficients" (related to J) of the weights of the θ -equivariant stable complex structure. In that way, it provides information which vectors $\{a_1(w) \cdot w(\alpha_1), \dots, a_n(w) \cdot w(\alpha_n)\}$, $w \in W_G/W_H$ can not be realized as the weight vectors of some θ equivariant stable complex structure.

Example 13. Consider complex projective space $\mathbb{C}P^3$ and let c_τ be a stable complex structure on $\mathbb{C}P^3$ equivariant under the canonical action of the torus T^4 . By Corollary 7, the weights for c_τ at the fixed points are $a_1(w) \cdot w(x_1 - x_4), a_2(w) \cdot w(x_2 - x_4), a_3(w) \cdot w(x_3 - x_4)$, where $w \in W_{U(4)}/W_{U(3)} = \mathbb{Z}_4$. Corollary 9 implies that the coefficients in t and t^2 in the polynomial

$$(34) \quad \prod_{1 \leq i < j \leq 4} (x_i - x_j) \cdot \sum_{w \in \mathbb{Z}_4} \frac{f(ta_1(w)w(x_1 - x_4))f(ta_2(w)w(x_2 - x_4))f(ta_3(w)w(x_3 - x_4))}{w(x_1 - x_4)w(x_2 - x_4)w(x_3 - x_4)},$$

where $f(t) = 1 + a_1t + a_2t^2 + a_3t^3$, have to be zero. The coefficient in t for (34) is a polynomial $P(x_1, x_2, x_3, x_4)$ of degree 4 whose coefficients are some linear combinations of the numbers $a_i(w)$. Some of them are as follows

$$\begin{aligned} x_1x_4^3 : a_2(2) - a_3(3), \quad x_1^3x_4 : a_1(3) - a_1(2), \quad x_1^3x_2 : a_1(0) - a_1(3), \quad x_2^3x_4 : a_2(1) - a_2(3), \\ x_3x_4^3 : a_1(1) - a_2(2), \quad x_3^3x_4 : a_3(2) - a_3(1), \quad x_1x_3^3 : a_3(0) - a_3(2), \end{aligned}$$

what implies that

$$(35) \quad \begin{aligned} a_1(0) = a_1(3) = a_1(2), \quad a_2(0) = a_2(1) = a_2(3), \\ a_3(0) = a_3(1) = a_3(2), \quad a_1(1) = a_2(2) = a_3(3). \end{aligned}$$

The direct computation shows that the requirement $P(x_1, x_2, x_3, x_4) \equiv 0$ is equivalent to the relations (35) and also that the same relations give that the coefficient in t^2 in the polynomial (34) will be zero.

Therefore, the weights for c_τ at the fixed points are completely determined by the values for $a_1(0), a_2(0), a_3(0), a_1(1)$. By Corollary 8 the signs at the fixed points are also determined by these values up to factor $\epsilon = \pm 1$ depending if c_τ and standard canonical structure on $\mathbb{C}P^3$ give the same orientation or not.

It turns out that in this case for all possible values ± 1 for $a_1(0), a_2(0), a_3(0), a_1(1)$ the corresponding vectors (30) can be realized as the weights vectors of the stable complex structures. In order to verify this we look at the decomposition $T(\mathbb{C}P^3) \oplus \mathbb{C} \cong \hat{\eta} \oplus \hat{\eta} \oplus \hat{\eta} \oplus \hat{\eta}$, where $\hat{\eta}$ is the

conjugate to the Hopf bundle over \mathbb{CP}^3 . Using this decomposition we can get the stable complex structures on $T(\mathbb{CP}^3)$ by choosing the complex structures on each $\hat{\eta}$. The weight vector for any such stable complex structure is determined by the corresponding choices for the values of $a_1(0), a_2(0), a_3(0), a_1(1)$. For example if we take $a_1(0) = a_2(0) = a_3(0) = a_1(1) = 1$ we get the weights for the standard complex structure which we can get from the stable complex structure $\hat{\eta} \oplus \hat{\eta} \oplus \hat{\eta} \oplus \hat{\eta}$.

If we take $a_1(0) = a_2(0) = a_3(0) = 1, a_1(1) = -1$ the corresponding weights come from the stable complex structure $\hat{\eta} \oplus \hat{\eta} \oplus \hat{\eta} \oplus \eta$. It follows that this stable complex structure and the standard complex structure define an opposite orientations on \mathbb{CP}^3 and, therefore, $\epsilon = -1$. Using (31) we obtain that related to this stable complex structure the signs at the fixed points are $\text{sign}(0) = -1, \text{sign}(1) = \text{sign}(2) = \text{sign}(3) = 1$. The weights at the fixed points are:

$$(0) : x_1 - x_4, x_2 - x_4, x_3 - x_4; \quad (1) : x_1 - x_4, x_2 - x_1, x_3 - x_1 ;$$

$$(2) : x_1 - x_2, x_2 - x_4, x_3 - x_2; \quad (3) : x_1 - x_3, x_2 - x_3, x_3 - x_4 .$$

By Example 6 we obtain that the number s_3 for \mathbb{CP}^3 related to this stable complex structure is equal to -2 . In that way it shows that \mathbb{CP}^3 with this non-standard stable complex structure realizes multiplicative generator in complex cobordism ring of dimension 6.

The relations (35) also give an examples of the vectors (30) that can not be realized as the weight vectors of the stable complex structures on \mathbb{CP}^3 equivariant under the canonical torus action.

Example 14. Using Proposition 7, we can also find an examples of the vectors (30) on the flag manifold $U(3)/T^3$ or Grassmann manifold $G_{4,2}$ that can *not* be realized as the weight vectors of the stable complex structures.

In a case of $U(3)/T^3$ any stable complex structure has the weights of the form $a_1(\sigma)\sigma(x_1 - x_2), a_2(\sigma)\sigma(x_1 - x_3), a_3(\sigma)\sigma(x_2 - x_3)$, where $\sigma \in S_3$. The relation (22) gives that for $k = 1$ the numbers $a_i(\sigma)$ have to satisfy the following relation

$$\sum_{\sigma \in S_3} \frac{a_1(\sigma)\sigma(x_1 - x_2) + a_2(\sigma)\sigma(x_1 - x_3) + a_3(\sigma)\sigma(x_2 - x_3)}{\sigma(x_1 - x_2)\sigma(x_1 - x_3)\sigma(x_2 - x_3)} = 0 .$$

It is equivalent to

$$\begin{aligned} & a_1(123) + a_2(123) + a_1(213) - a_3(213) + a_2(321) + a_3(321) \\ & - a_1(132) - a_2(132) - a_2(231) - a_3(231) - a_1(312) + a_3(312) = 0 , \\ & - a_1(123) + a_3(123) - a_1(213) - a_2(213) + a_1(321) - a_3(321) \\ & + a_2(132) - a_3(132) + a_1(231) + a_2(231) - a_2(312) - a_3(312) = 0 . \end{aligned}$$

It follows from these relations that the vector (30) determined with $a_1(123) = -1$ and $a_i(\sigma) = 1$ for all the others $1 \leq i \leq 3$ and $\sigma \in S_3$, can not be obtained as the weight vector of some stable complex structure on $U(3)/T^3$ equivariant under the canonical action of the maximal torus.

Using the same argument we can also conclude that for the Grassmannian $G_{4,2}$ the vector determined with $a_1(1234) = -1$ and $a_i(\sigma) = 1$ for all the others $1 \leq i \leq 4$ and $\sigma \in S_4/S_2 \times S_2$ can not be realized as the weight vector of the stable complex structure equivariant under the canonical action of T^4 .

8. SOME APPLICATIONS.

8.1. Flag manifolds $U(n)/T^n$. We consider invariant complex structure on $U(n)/T^n$. Recall [1] that the Weyl group $W_{U(n)}$ is the symmetric group and it permutes the coordinates x_1, \dots, x_n on Lie algebra \mathfrak{t}^n for T^n . The canonical action of the torus T^n on this manifold has $\|W_{U(n)}\| = \chi(U(n)/T^n) = n!$ fixed points and its weights at identity point are given by the roots of $U(n)$.

We first consider the case $n = 3$ and apply our results to explicitly compute cobordism class and Chern numbers for $U(3)/T^3$. The roots for $U(3)$ are $x_1 - x_2$, $x_1 - x_3$ and $x_2 - x_3$. Therefore the cobordism class for $U(3)/T^3$ is given as the coefficient for t^3 in the polynomial

$$[U(3)/T^3] = \sum_{\sigma \in S_3} \sigma \left(\frac{f(t(x_1 - x_2))f(t(x_1 - x_3))f(t(x_2 - x_3))}{(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)} \right),$$

where $f(t) = 1 + a_1 t + a_2 t^2 + a_3 t^3$, what implies

$$[U(3)/T^3] = 6(a_1^3 + a_1 a_2 - a_3).$$

This gives that the characteristic numbers s_ω for $U(3)/T^3$ are

$$s_{(3,0,0)} = 6, \quad s_{(1,1,0)} = 6, \quad s_{(0,0,1)} = -6.$$

By Theorem 7 we have the following relations between characteristic numbers s_ω and Chern numbers c^ω

$$c_3 = 6, \quad c_1 c_2 - 3c_3 = 6, \quad c_1^3 - 3c_1 c_2 + 3c_3 = -6, \quad \text{what gives } c_1 c_2 = 24, \quad c_1^3 = 48.$$

To simplify the notations we take further $\Delta_n = \prod_{1 \leq i < j \leq n} (x_i - x_j)$.

Theorem 8. *The Chern-Dold character of the toric genus for the flag manifold $U(n)/T^n$ is given by the formula:*

$$(36) \quad ch_U \Phi(U(n)/T^n) = \frac{1}{\Delta_n} \sum_{\sigma \in S_n} \text{sign}(\sigma) \sigma \left(\prod_{1 \leq i < j \leq n} f(x_i - x_j) \right),$$

where $f(t) = 1 + \sum_{i \geq 1} a_i t^i$ and $\text{sign}(\sigma)$ is the sign of the permutation σ .

8.1.1. Using of divided difference operators. Consider the ring of the symmetric polynomials $\text{Sym}_n \subset \mathbb{Z}[x_1, \dots, x_n]$. There is a linear operator (see [25])

$$L : \mathbb{Z}[x_1, \dots, x_n] \longrightarrow \text{Sym}_n : L\mathbf{x}^\xi = \frac{1}{\Delta_n} \sum_{\sigma \in S_n} \text{sign}(\sigma) \sigma(\mathbf{x}^\xi),$$

where $\xi = (j_1, \dots, j_n)$ and $\mathbf{x}^\xi = x_1^{j_1} \dots x_n^{j_n}$.

It follows from the definition of Schur polynomials $\text{Sh}_\lambda(x_1, \dots, x_n)$ where $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0)$ (see [25]), that

$$L\mathbf{x}^{\lambda+\delta} = \text{Sh}_\lambda(x_1, \dots, x_n),$$

where $\delta = (n-1, n-2, \dots, 1, 0)$ and $L\mathbf{x}^\delta = 1$. Moreover, the operator L have the following properties:

- $L\mathbf{x}^\xi = 0$, if $j_1 \geq j_2 \geq \dots \geq j_n \geq 0$ and $\xi \neq \lambda + \delta$ for some $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0)$;
- $L\mathbf{x}^\xi = \text{sign}(\sigma) L\sigma(\mathbf{x}^\xi)$, where $\xi = (j_1, \dots, j_n)$ and $\sigma\xi = \xi'$, where $\sigma \in S_n$ and $\xi' = (j'_1 \geq j'_2 \geq \dots \geq j'_n \geq 0)$;

- L is a homomorphism of Sym_n -modules.

We have

$$(37) \quad \prod_{1 \leq i < j \leq n} f(t(x_i - x_j)) = 1 + \sum_{|\xi| > 0} P_\xi(a_1, \dots, a_n, \dots) t^{|\xi|} \mathbf{x}^\xi,$$

where $|\xi| = \sum_{q=1}^n j_q$.

Corollary 10. *The Chern-Dold character of the toric genus for the flag manifold $U(n)/T^n$ is given by the formula:*

$$ch_U \Phi(U(n)/T^n) = \sum_{|\lambda| \geq 0} \left(\sum_{\sigma \in S_n} \text{sign}(\sigma) P_{\sigma(\lambda+\delta)}(a_1, \dots, a_n, \dots) \right) \text{Sh}_\lambda(x_1, \dots, x_n),$$

where $\delta = (n-1, n-2, \dots, 1, 0)$, $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0)$. In particular,

$$(38) \quad [U(n)/T^n] = \sum_{\sigma \in S_n} \text{sign}(\sigma) P_{\sigma\delta}(a_1, \dots, a_n, \dots).$$

Proof. Set $m = \frac{n(n-1)}{2}$. From Theorem 8 and the formula (37) we obtain:

$$ch_U \Phi(U(n)/T^n) = \sum_{|\xi| \geq m} P_\xi L \mathbf{x}^\xi.$$

The first property of the operator L gives that for any ξ we will have $L \mathbf{x}^\xi = 0$, whenever $\mathbf{x}^\xi \neq \sigma(\mathbf{x}^{\lambda+\delta})$ for some $\sigma \in S_n$ and $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0)$. The second property gives that $L \sigma(\mathbf{x}^{\lambda+\delta}) = \text{sign}(\sigma) \text{Sh}_\lambda(x_1, \dots, x_n)$. \square

Remark 16. 1. In the case $n = 2$ this corollary gives the result of Example 1.

2. As we will show in Corollary 14 below, polynomials $P_{\sigma\delta}$ in the formula (38) appears to be polynomials only in variables a_1, \dots, a_{2n-3} .

Set by definition

$$1 + \sum_{|\xi| > 0} \sigma^{-1}(P_\xi) t^{|\xi|} \mathbf{x}^\xi = \sigma \left(\prod_{1 \leq i < j \leq n} f(t(x_i - x_j)) \right),$$

where $\sigma \in S_n$ on the right acts by the permutation of variables x_1, \dots, x_n . Directly from the definition we have

$$1 + \sum_{|\xi| > 0} \sigma^{-1}(P_\xi) t^{|\xi|} \mathbf{x}^\xi = 1 + \sum_{|\xi| > 0} P_\xi t^{|\xi|} \sigma(\mathbf{x}^\xi).$$

Therefore

$$(39) \quad \sigma(P_\xi) = P_{\sigma\xi}.$$

Together with Corollary 10 the formula (39) implies the following Theorem.

Theorem 9.

$$(40) \quad [U(n)/T^n] = \left(\sum_{\sigma \in S_n} \text{sign}(\sigma) \sigma \right) P_\delta(a_1, \dots, a_n, \dots).$$

Corollary 11.

$$\sigma[U(n)/T^n] = \text{sign}(\sigma) [U(n)/T^n].$$

Proof.

$$\begin{aligned}\sigma[U(n)/T^n] &= \left(\sum_{\tilde{\sigma} \in S_n} \text{sign}(\tilde{\sigma}) \sigma \tilde{\sigma} \right) P_\delta(a_1, \dots, a_n, \dots) = \\ &= \left(\sum_{\bar{\sigma} \in S_n} \text{sign}(\sigma^{-1} \bar{\sigma}) \bar{\sigma} \right) P_\delta(a_1, \dots, a_n, \dots) = \text{sign}(\sigma^{-1}) \left(\sum_{\bar{\sigma} \in S_n} \text{sign}(\bar{\sigma}) \bar{\sigma} \right) P_\delta(a_1, \dots, a_n, \dots) .\end{aligned}$$

Since $\text{sign}(\sigma) = \text{sign}(\sigma^{-1})$, the formula follows. \square

Example 15. The direct computation gives that for $n = 3$ the polynomials $P_{\sigma\delta}(a_1, a_2, a_3) = \sigma(P_\delta)$ from (37), where $\delta = (2, 1, 0)$ and $\sigma \in S_3$ are

$$\begin{aligned}P_\delta &= a_1^3 - a_1 a_2 - 3a_3 , \\ (12)P_\delta &= (13)P_\delta = -P_\delta , \\ (23)P_\delta &= -(a_1^3 + 5a_1 a_2 + 3a_3) .\end{aligned}$$

The action of the transpositions implies that the rest two permutations act as

$$(312)P_\delta = P_\delta, \quad (231)P_\delta = -(23)P_\delta .$$

Using Corollary 10 we obtain the cobordism class $[U(3)/T^3] = 6(a_1^3 + a_1 a_2 - a_3)$. The above also gives that the symmetric group S_3 acts non-trivially on $P_\delta(a_1, a_2, a_3)$ and $\sum_{\sigma \in S_3} \sigma P_\delta(a_1, a_2, a_3) = 0$.

We have

$$\prod_{1 \leq i < j \leq n} f(t(x_i - x_j)) \equiv \prod_{1 \leq i < j \leq n} f(t(x_i + x_j)) \pmod{2} .$$

Using that $\prod_{1 \leq i < j \leq n} f(t(x_i + x_j))$ is the symmetric series of variables x_1, \dots, x_n we obtain

$$\sigma(P_\xi) \equiv P_\xi \pmod{2} ,$$

for any $\sigma \in S_n$. Thus Theorem 9 implies the following:

Corollary 12. All Chern numbers of the manifold $U(n)/T^n$, $n \geq 2$, are even.

Remark 17. Cobordism class $[U(3)/T^3]$ gives nonzero element in $\Omega_U^{-6} \otimes \mathbb{Z}/2$ because $s_3(U(3)/T^3) \equiv 2 \pmod{4}$.

The characteristic number s_m for $U(n)/T^n$ is given as

$$(41) \quad s_m(U(n)/T^n) = \sum_{1 \leq i < j \leq n} L(x_i - x_j)^m .$$

Corollary 10 implies the following:

Corollary 13. $s_1(U(2)/T^2) = 2$; $s_3(U(3)/T^3) = -6$ and

$$(42) \quad s_m(U(n)/T^n) = 0 ,$$

where $m = \frac{n(n-1)}{2}$ and $n > 3$.

We can push up this further. Denote by $(u_1, \dots, u_m) = ((x_i - x_j), i < j)$, where $m = \frac{n(n-1)}{2}$. Then for $\omega = (i_1, \dots, i_m)$, $\|\omega\| = m$ we have that

$$O\left((u_1 \cdots u_{i_1})(u_{i_1+1}^2 \cdots u_{i_1+i_2}^2) \cdots (u_{i_1+\dots+i_{m-1}+1}^m \cdots u_{i_1+\dots+i_m}^m)\right) = \sum_{|\xi|=m} \alpha_{\omega,\xi} \mathbf{x}^\xi.$$

This implies that

$$(43) \quad s_\omega(U(n)/T^n) = \sum_{|\xi|=m} \alpha_{\omega,\xi} L \mathbf{x}^\xi = \sum_{\sigma \in S_n} \text{sign}(\sigma) \alpha_{\omega,\sigma\delta}.$$

Therefore, if $\xi = (j_1, \dots, j_n)$, then $\max_{p_1, \dots, p_s} (j_{p_1} + \dots + j_{p_s}) = s \left(n - \frac{s+1}{2} \right)$, $1 \leq s \leq n$. In particular, it holds that $\max_{p_1, p_2} (j_{p_1} + j_{p_2}) = 2n - 3$.

Corollary 14. *Let $\omega = (i_1, \dots, i_m)$ such that $i_k \neq 0$ for some $k > 2n - 3$, then*

$$s_\omega(U(n)/T^n) = 0.$$

If $\omega = (i_1, \dots, i_k)$, $\|\omega\| = m$, does not satisfy Corollary 14, but $i_{k_1}, \dots, i_{k_l} \neq 0$ for some k_1, \dots, k_l then we have that $k_p = 2(n-1) - q_p$, for $q_p \geq 1$, $1 \leq p \leq l$. In this case we can say the following.

Corollary 15. *If $n \geq 2l$ and $\sum_{p=1}^l q_p < l(2l-1)$ then*

$$s_\omega(U(n)/T^n) = 0.$$

Remark 18. From the second property of the operator L we obtain that $L\mathcal{P}(x_1, \dots, x_n) = 0$ for any series $\mathcal{P}(x_1, \dots, x_n)$, whenever $\sigma(\mathcal{P}(x_1, \dots, x_n)) = \varepsilon \mathcal{P}(x_1, \dots, x_n)$ for a permutation $\sigma \in S_n$, where $\varepsilon = \pm 1$ and $\varepsilon \cdot \text{sign}(\sigma) = -1$. This, in particular, gives that $L(\mathcal{P}(x_1, \dots, x_n) + \sigma_{ij}(\mathcal{P}(x_1, \dots, x_n))) = 0$ for any transposition σ_{ij} of x_i and x_j , where $1 \leq i < j \leq n$.

Using Remark 18 we can compute some more characteristic numbers of the flag manifolds.

Corollary 16. *Let $n = 4q$ or $4q + 1$ and $\omega = (i_1, \dots, i_m)$, $\|\omega\| = m$, where $i_{2l-1} = 0$ for $l = 1, \dots, \frac{m}{2}$. Then $s_\omega(U(n)/T^n) = 0$.*

Since $\sigma_{12}((x_1 - x_2)^{2l} \prod_{\substack{1 \leq i < j \leq n \\ (i,j) \neq (1,2)}} f(t(x_i - x_j))) = (x_1 - x_2)^{2l} \prod_{\substack{1 \leq i < j \leq n \\ (i,j) \neq (1,2)}} f(t(x_i - x_j))$ we have, also

because of Remark 18, that

$$L\left(\prod_{1 \leq i < j \leq n} f(t(x_i - x_j))\right) = L\left(\tilde{f}(t(x_1 - x_2)) \prod_{\substack{1 \leq i < j \leq n \\ (i,j) \neq (1,2)}} f(t(x_i - x_j))\right),$$

where $\tilde{f}(t) = \sum_{l \geq 1} a_{2l-1} t^{2l-1}$. Using this property of L once more we obtain

Theorem 10. *For $n \geq 4$ the cobordism class for the flag manifold $U(n)/T^n$ is given as the coefficient for $t^{\frac{n(n-1)}{2}}$ in the series in t*

$$(44) \quad L\left(\tilde{f}(t(x_1 - x_2)) \tilde{f}(t(x_{n-1} - x_n)) \prod_{\substack{1 \leq i < j \leq n \\ (i,j) \neq (1,2), (n-1,n)}} f(t(x_i - x_j))\right).$$

Remark 19. Corollary 15 implies that if $s_\omega \neq 0$ for some $\omega = (i_1, \dots, i_m)$, then for some $1 \leq l \leq \frac{m}{2}$ it has to be $i_{2l-1} \neq 0$. The Theorem 10 gives stronger results that, for $n \geq 4$ in polynomials $P_{\sigma\delta}$ in (38) each monom contains the factor $a_{2i_1-1}a_{2i_2-1}$.

Theorem 10 provide a way for direct computation of the number s_ω , for $\omega = (i_1, \dots, i_m)$ such that $\|\omega\| = 2$, where $\|\omega\| = i_1 + \dots + i_m$. For $n > 5$ we have that $s_\omega(U(n)/T^n) = 0$ for such ω . For $n = 4$ and $n = 5$ these numbers can be computed very straightforward as the next example shows.

Example 16. We provide computation of the characteristic number $s_{(1,0,0,0,1,0)}$ for $U(4)/T^4$. From the formula (44) we obtain immediately:

$$\begin{aligned} s_{(1,0,0,0,1,0)}(U(4)/T^4) &= L\left((x_1 - x_2)(x_3 - x_4)^5 + (x_1 - x_2)^5(x_3 - x_4)\right) = \\ &= 10L\left((x_1 - x_2)(x_3 - x_4)(x_1^2x_2^2 + x_3^2x_4^2)\right) = \\ &= 20L\left(x_1^3x_2^2(x_3 - x_4) + (x_1 - x_2)x_3^3x_4^2\right) = \\ &= 40L\left(x_1^3x_2^2x_3 + x_1x_3^3x_4^2\right) = 80. \end{aligned}$$

Remark 20. We want to emphasize that the formula (36) gives the description of the cobordism classes of the flag manifolds in terms of *divided difference operators*. The divided difference operators are defined with (see [3])

$$\partial_{ij}P(x_1, \dots, x_n) = \frac{1}{x_i - x_j} \left(P(x_1, \dots, x_n) - \sigma_{ij}P(x_1, \dots, x_n) \right),$$

where $i < j$. Put $\sigma_{i,i+1} = \sigma_i$, $\partial_{i,i+1} = \partial_i$, $1 \leq i \leq n-1$. We can write down operator L as the following composition (see [14, 24])

$$L = (\partial_1\partial_2 \cdots \partial_{n-1})(\partial_1\partial_2 \cdots \partial_{n-2}) \cdots (\partial_1\partial_2)\partial_1.$$

Denote by w_0 the permutation $(n, n-1, \dots, 1)$. Wright down a permutation $w \in S_n$ in the form $w = w_0\sigma_{i_1} \cdots \sigma_{i_p}$ and set $\nabla_w = \partial_{i_p} \cdots \partial_{i_1}$. It is natural to set $\nabla_{w_0} = I$ — identity operator. The space of operators ∇_w is dual to the space of the Schubert polynomials $\mathfrak{S}_w = \mathfrak{S}_w(x_1, \dots, x_n)$, since it follows from their definition that $\mathfrak{S}_w = \nabla_w \mathbf{x}^\delta$. Note that $\mathfrak{S}_{w_0} = \mathbf{x}^\delta$. For the identity permutation $e = (1, 2, \dots, n)$ we have $e = w_0 \cdot w_0^{-1}$. So $\nabla_e = L$ and $\mathfrak{S}_e = \nabla_e \mathbf{x}^\delta = 1$.

Schubert polynomials were introduced in [3] and in [13] in context of an arbitrary root systems. The main reference on algebras of operators ∇_w and Schubert polynomials \mathfrak{S}_w is [24].

The description of the cohomology rings of the flag manifolds $U(n)/T^n$ and Grassmann manifolds $G_{n,k} = U(n)/(U(k) \times U(n-k))$ in the terms of Schubert polynomials is given in [14].

The description of the *complex cobordism ring* of the flag manifolds G/T , for G compact, connected Lie group and T its maximal torus, in the terms of the Schubert polynomials calculus is given in [9, 10].

8.2. Grassmann manifolds. As a next application we will compute cobordism class, characteristic numbers s_ω and, consequently, Chern numbers for invariant complex structure on Grassmannian $G_{4,2} = U(4)/(U(2) \times U(2)) = SU(4)/S(U(2) \times U(2))$. Note that, it follows by [8] that, up to equivalence, $G_{4,2}$ has one invariant complex structure J . The corresponding Lie algebra description for $G_{4,2}$ is $A_3/(\mathfrak{t}^1 \oplus A_1 \oplus A_1)$.

The number of the fixed points under the canonical action of T^3 on $G_{4,2}$ is, by Theorem 3, equal to 6. Let x_1, x_2, x_3, x_4 be canonical coordinates on maximal abelian algebra for A_3 . Then x_1, x_2 and x_3, x_4 represents canonical coordinates for $A_1 \oplus A_1$. The weights of this action at identity point $(T_e(G_{4,2}), J)$ are given by the positive complementary roots $x_1 - x_3, x_1 - x_4, x_2 - x_3, x_2 - x_4$ for A_3 related to $A_1 \oplus A_1$ that define J .

The Weyl group $W_{U(4)}$ is the symmetric group of permutation on coordinates x_1, \dots, x_4 and the Weyl group $W_{U(2) \times U(2)} = W_{U(2)} \times W_{U(2)}$ is the product of symmetric groups on coordinates x_1, x_2 and x_3, x_4 respectively. Let $w_j \in W_{U(4)}/W_{U(2) \times U(2)}$. Corollary 2 gives that the cobordism class $[G_{4,2}]$ is the coefficient for t^4 in polynomial

$$(45) \quad \sum_{j=1}^6 w_j \left(\frac{f(t(x_1 - x_3))f(t(x_1 - x_4))f(t(x_2 - x_3))f(t(x_2 - x_4))}{(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)} \right) = \\ = \frac{1}{4} L \left((x_1 - x_2)(x_3 - x_4) f(t(x_1 - x_3)) f(t(x_1 - x_4)) f(t(x_2 - x_3)) f(t(x_2 - x_4)) \right),$$

where $f(t) = 1 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4$.

Expanding formula (45) we get that

$$(46) \quad [G_{4,2}] = 2(3a_1^4 + 12a_1^2 a_2 + 7a_2^2 + 2a_1 a_3 - 10a_4).$$

The characteristic numbers s_ω can be read off from this formula:

$$s_{(4,0,0,0)} = 6, \quad s_{(2,1,0,0)} = 24, \quad s_{(0,2,0,0)} = 14, \quad s_{(1,0,1,0)} = 4, \quad s_{(0,0,0,1)} = -20.$$

The coefficients $\beta_{\omega\xi}$ from Theorem 7 can be explicitly computed and for 8-dimensional manifold give the following relation between characteristic numbers s_ω and Chern numbers:

$$s_{(0,0,0,1)} = c_1^4 - 4c_1^2 c_2 + 2c_2^2 + 4c_1 c_3 - 4c_4, \quad s_{(2,1,0,0)} = c_1 c_3 - 4c_4, \\ s_{(0,2,0,0)} = c_2^2 - 2c_1 c_3 + 2c_4, \quad s_{(1,0,1,0)} = c_1^2 c_2 - c_1 c_3 + 4c_4 - 2c_2^2, \quad s_{(4,0,0,0)} = c_4.$$

We deduce that the Chern numbers for $(G_{4,2}, J)$ are

$$c_4 = 6, \quad c_1 c_3 = 48, \quad c_2^2 = 98, \quad c_1^2 c_2 = 224, \quad c_1^4 = 512.$$

The given example generalizes as follows. Denote by $\Delta_{p,q} = \prod_{p \leq i < j \leq q} (x_i - x_j)$, then $\Delta_n = \Delta_{1,n}$.

Theorem 11. *The cobordism class for Grassmann manifold $G_{q+l,l}$ is given as the coefficient for t^{lq} in the series in t*

$$(47) \quad \sum_{\sigma \in S_{q+l}/S_q \times S_l} \sigma \left(\prod \frac{f(t(x_i - x_j))}{(x_i - x_j)} \right) = \frac{1}{q!l!} L \left(\Delta_q \Delta_{q+1,q+l} \prod f(t(x_i - x_j)) \right),$$

where $1 \leq i \leq q$, $(q+1) \leq j \leq (q+l)$ and S_{q+l} is the symmetric group.

Let us introduce the polynomials $Q_{(q+l,l)\xi}$ defined with

$$\Delta_q \Delta_{q+1,q+l} \prod_{\substack{1 \leq i \leq q \\ q+1 \leq j \leq q+l}} f(t(x_i - x_j)) = \sum_{|\xi| \geq \frac{(q+l)^2 - (q+l)}{2}} Q_{(q+l,l)\xi}(a_1, \dots, a_n, \dots) t^{|\xi| - \frac{(q+l)^2 - (q+l)}{2}} x^\xi,$$

where $\xi = (j_1, \dots, j_{q+l})$ and $|\xi| = \sum_{k=1}^{q+l} j_k$. Appealing to Theorem 11 we obtain the following.

Corollary 17. *The cobordism class for Grassmann manifold $G_{q+l,l}$ is given with*

$$(48) \quad [G_{q+l,l}] = \frac{1}{q!l!} \sum_{\sigma \in S_{q+l}} \text{sign}(\sigma) Q_{(q+l,l)\sigma\delta}(a_1, \dots, a_{ql}) ,$$

where $\delta = (q + l - 1, q + l - 2, \dots, 0)$.

Example 17. For $q = l = 2$ the calculations give that the polynomials $Q_{\sigma\delta} = Q_{(4,2)\sigma\delta}(a_1, a_2, a_3, a_4)$ are as follows

$$\begin{aligned} Q_{(3,2,1,0)} &= -Q_{(2,3,1,0)} = -Q_{(3,2,0,1)} = Q_{(2,3,0,1)} = Q_{(1,0,3,2)} = -Q_{(1,0,2,3)} = -Q_{(0,1,3,2)} = Q_{(0,1,2,3)} \\ &= a_1^4 + 4a_2^2 - 4a_1a_3 , \\ Q_{(2,1,3,0)} &= -Q_{(1,2,3,0)} = -Q_{(2,1,0,3)} = Q_{(1,2,0,3)} = Q_{(3,0,2,1)} = -Q_{(3,0,1,2)} = -Q_{(0,3,2,1)} = Q_{(0,3,1,2)} \\ &= a_1^4 + 4a_1^2a_2 + a_2^2 + 6a_1a_3 - 6a_4 , \\ Q_{(1,3,2,0)} &= -Q_{(3,1,2,0)} = Q_{(3,1,0,2)} = -Q_{(1,3,0,2)} = Q_{(2,0,1,3)} = -Q_{(2,0,3,1)} = Q_{(0,2,3,1)} = -Q_{(0,2,1,3)} \\ &= a_1^4 + 8a_1^2a_2 + 2a_2^2 - 4a_4 , \end{aligned}$$

Using Corollary 17 we obtain the formula (46) for the cobordism class $[G_{4,2}]$.

8.3. Homogeneous space $SU(4)/S(U(1) \times U(1) \times U(2))$. Following [8] and [20] we know that 10-dimensional space $M^{10} = SU(4)/S(U(1) \times U(1) \times U(2))$ admits, up to equivalence, two invariant complex structure J_1 and J_2 and one non-integrable invariant almost complex structure J_3 . We provide here the description of cobordism classes for all of three invariant almost complex structures. The Chern numbers for all the invariant almost complex structures are known and they have been completely computed in [20] through multiplication in cohomology. We provide also their computation using our method.

The corresponding Lie algebra description for M^{10} is $A_3/(\mathfrak{t}^2 \oplus A_1)$. Let x_1, x_2, x_3, x_4 be canonical coordinates on maximal Abelian subalgebra for A_3 . Then x_1, x_2 represent canonical coordinates for A_1 . The number of fixed points under the canonical action of T^3 on M^{10} is, by Theorem 3, equal to 12.

8.3.1. The invariant complex structure J_1 . The weights of the action of T^3 on M^{10} at identity point related to J_1 are given by the complementary roots $x_1 - x_3, x_1 - x_4, x_2 - x_3, x_2 - x_4, x_3 - x_4$ for A_3 related to A_1 , (see [8], [20]). The cobordism class $[M^1, J_1]$ is, by Corollary 2, given as the coefficient for t^5 in polynomial

$$\sum_{j=1}^{12} w_j \left(\frac{f(t(x_1 - x_3))f(t(x_1 - x_4))f(t(x_2 - x_3))f(t(x_2 - x_4))f(t(x_3 - x_4))}{(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)(x_3 - x_4)} \right) ,$$

where $w_j \in W_{U(4)}/W_{U(2)}$ and $f(t) = 1 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5$.

Therefore we get that

$$[M^{10}, J_1] = 4(3a_1^5 + 12a_1^3a_2 + 7a_1a_2^2 - 5a_1^2a_3 - 2a_2a_3 - 10a_1a_4 + 5a_5) .$$

By Theorem 7 we get the following relations between characteristic numbers s_ω and Chern numbers for (M^{10}, J_1) .

$$\begin{aligned} s_{(0,0,0,0,1)} &= 20 = c_1^5 - 5c_1^3c_2 + 5c_1^2c_3 + 5c_1c_2^2 - 5c_1c_4 - 5c_2c_3 + 5c_5 , \\ s_{(1,2,0,0,0)} &= 28 = c_2c_3 - 3c_1c_4 + 5c_5 , \quad s_{(2,0,1,0,0)} = -20 = c_1^2c_3 - c_1c_4 - 2c_2c_3 + 5c_5 , \\ s_{(0,1,1,0,0)} &= -8 = -2c_1^2c_3 + c_1c_2^2 - c_2c_3 + 5c_1c_4 - 5c_5 , \quad s_{(3,1,0,0,0)} = 48 = c_1c_4 - 5c_5 , \end{aligned}$$

$$s_{(1,0,0,1,0)} = -40 = c_1^3 c_2 - c_1^2 c_3 - 3c_1 c_2^2 + c_1 c_4 + 5c_2 c_3 - 5c_5, \quad s_{(5,0,0,0,0)} = 12 = c_5.$$

This implies that the Chern numbers for (M^{10}, J_1) are as follows:

$$c_5 = 12, \quad c_1 c_4 = 108, \quad c_2 c_3 = 292, \quad c_1^2 c_3 = 612, \quad c_1 c_2^2 = 1028, \quad c_1^3 c_2 = 2148, \quad c_1^5 = 4500.$$

8.3.2. The invariant complex structure J_2 . The weights of the action of T^3 on M^{10} at identity point related to J_2 are given by the positive complementary roots $x_4 - x_1, x_4 - x_2, x_4 - x_3, x_1 - x_3, x_2 - x_3$ for A_3 related to A_1 , (see [8], [20]). The cobordism class $[M^1, J_2]$ is, by Corollary 2, given as the coefficient for t^5 in polynomial

$$\sum_{j=1}^{12} w_j \left(\frac{f(t(x_4 - x_1))f(t(x_4 - x_2))f(t(x_4 - x_3))f(t(x_1 - x_3))f(t(x_2 - x_3))}{(x_4 - x_1)(x_4 - x_2)(x_4 - x_3)(x_1 - x_3)(x_2 - x_3)} \right),$$

where $w_j \in W_{U(4)}/W_{U(2)}$ and $f(t) = 1 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5$.

Therefore we get that

$$[M^{10}, J_2] = 4(3a_1^5 + 12a_1^3 a_2 + 7a_1 a_2^2 - 5a_1^2 a_3 + 8a_2 a_3 - 10a_1 a_4 - 5a_5).$$

Remark 21. We could proceed with the description of cobordism class $[M^{10}, J_2]$ appealing to the description of J_1 from 8.3.1 and applying the results from Section 7. The corresponding description of the weights for the action on T^3 on (M^{10}, J_2) looks like

$$(+1) \cdot (x_1 - x_3), (-1) \cdot (x_1 - x_4), (+1) \cdot (x_2 - x_3), (-1) \cdot (x_2 - x_4), (-1) \cdot (x_3 - x_4),$$

what means that

$$a_1(e) = +1, a_2(e) = -1, a_3(e) = +1, a_4(e) = -1, a_5(e) = -1.$$

Since J_1 and J_2 define on M^{10} an opposite orientation, we have that $\epsilon = -1$ and it follows that the fixed points have sign $+1$.

Applying the same procedure as above we get that the Chern numbers for (M^{10}, J_2) are:

$$c_5 = 12, \quad c_1 c_4 = 108, \quad c_2 c_3 = 292, \quad c_1^2 c_3 = 612, \quad c_1 c_2^2 = 1068, \quad c_1^3 c_2 = 2268, \quad c_1^5 = 4860.$$

8.3.3. The invariant almost complex structure J_3 . The weights for the action of T^3 on M^{10} at identity point related to J_3 are given by complementary roots $x_1 - x_3, x_2 - x_3, x_4 - x_1, x_4 - x_2, x_3 - x_4$, (see [20]). Using Corollary 2 we get that the cobordism class for (M^{10}, J_3) is

$$[M^{10}, J_3] = 4(3a_1^5 - 12a_1^3 a_2 + 7a_1 a_2^2 + 15a_1^2 a_3 - 12a_2 a_3 - 10a_1 a_4 + 15a_5).$$

Remark 22. We could also as in the previous case proceed with the computation of the cobordism class $[M^{10}, J_3]$ appealing on the on the description of J_1 from 8.3.1. The corresponding description of the weights is

$$(+1) \cdot (x_1 - x_3), (-1) \cdot (x_1 - x_4), (+1) \cdot (x_2 - x_3), (-1) \cdot (x_2 - x_4), (+1) \cdot (x_3 - x_4).$$

Since J_1 and J_2 define the same orientation it follows that all fixed points for an action T^3 on M^{10} have sign $+1$.

The characteristic numbers for (M^{10}, J_3) are given as coefficients in its cobordism class, what, as above, together with Theorem 7 gives that the Chern numbers for (M^{10}, J_3) are as follows:

$$c_5 = 12, \quad c_1 c_4 = 12, \quad c_2 c_3 = 4, \quad c_1^2 c_3 = 20, \quad c_1 c_2^2 = -4, \quad c_1^3 c_2 = -4, \quad c_1^5 = -20.$$

Remark 23. Further work on the studying of Chern numbers and the geometry for the generalizations of this example is done in [18] and in [20].

8.4. Sphere S^6 . According to [8] we know that the sphere $S^6 = G_2/SU(3)$ admits G_2 -invariant almost complex structure, but it does not admit G_2 -invariant complex structure. The existence of an invariant almost complex structure follows from the fact that $SU(3)$ is connected centralizer of an element of order 3 in G_2 which generates it's center, while the non-existence of an invariant complex structure is because the second Betti number for S^6 is zero. Note that being isotropy irreducible, $S^6 = G_2/SU(3)$ has unique, up to conjugation, invariant almost complex structure J .

The roots for the Lie algebra \mathfrak{g}_2 are given with (see [30])

$$\pm x_1, \pm x_2, \pm x_3, \pm(x_1 - x_2), \pm(-x_1 + x_3), \pm(-x_2 + x_3),$$

where $x_1 + x_2 + x_3 = 0$. It follows that the system of complementary roots for \mathfrak{g}_2 related to A_2 is $\pm x_1, \pm x_2, \pm x_3$. According to 4.1, since $S^6 = G_2/SU(3)$ is isotropy irreducible, this implies that the roots of an existing invariant almost complex structure J on S^6 are

$$\alpha_1 = x_1, \alpha_2 = x_2, \alpha_3 = x_3 = -(x_1 + x_2) .$$

The canonical action of a common maximal torus T^2 on $S^6 = G_2/SU(3)$ has $\chi(S^6) = 2$ fixed points.

By Theorem 3 we get that the weights at the fixed points for this action are given by the action of the Weil group W_{G_2} up to the action of the Weil group $W_{SU(3)}$ on the roots for J :

$$x_1, x_2, x_3 \text{ and } -x_1, -x_2, -x_3 .$$

Since the weights at these two fixed points are of opposite signs, Corollary 1 implies that in the Chern character of the universal toric genus for (S^6, J) the coefficients for a^ω are going to be zero for $\|\omega\|$ being even.

As the almost complex structure J is invariant under the action of the group G_2 , it follows by Lemma 1 that the universal toric genus for (S^6, J) belongs to the image of the ring $U^*(BG_2)$ in $U^*(BT^2)$ of the map induced by embedding $T^2 \subset G_2$. Furthermore, using Corollary 1 we obtain that the universal toric genus for (S^6, J) is series in σ_2 and σ_3 , where σ_2 and σ_3 are the elementary symmetric functions in three variables. The direct computation gives that the beginning terms in the series of the Chern character are

$$\begin{aligned} ch_U \Phi(S^6, J) = & 2(a_1^3 - 3a_1a_2 + 3a_3) + \\ & + 2(a_1a_2^2 - 2a_1^2a_3 - a_2a_3 + 5a_1a_4 - 5a_5)\sigma_2 + \\ & + 2(a_1a_3^2 - 2a_1a_2a_4 - a_3a_4 + 2a_1^2a_5 + 3a_2a_5 - 7a_1a_6 + 7a_7)\sigma_2^2 + \\ & + 2(3a_9 - 3a_1a_8 - 3a_2a_7 + 6a_3a_6 - 3a_4a_5 + 3a_1^2a_7 - 3a_1a_2a_6 - 3a_1a_3a_5 + \\ & + 3a_1a_4^2 + 3a_2^2a_5 - 3a_2a_3a_4 + a_3^3)\sigma_3^2 + \\ & + 2(-9a_9 + 9a_1a_8 - 5a_2a_7 + 3a_3a_6 - a_4a_5 - 2a_1^2a_7 + \\ & + 2a_1a_2a_6 - 2a_1a_3a_5 + a_1a_4^2)\sigma_2^3 + \dots \end{aligned}$$

In particular, we obtain that the cobordism class for (S^6, J) is

$$(49) \quad [S^6, J] = 2(a_1^3 - 3a_1a_2 + 3a_3) .$$

Remark 24. We can also compute cobordism class $[S^6, J]$ using relations between Chern numbers and characteristic numbers for an invariant almost complex structure given by Theorem 7:

$$c_3 = s_{(3,0,0)}, \quad c_1c_2 - 3c_3 = s_{(1,1,0)}, \quad c_1^3 - 3c_1c_2 + 3c_3 = s_{(0,0,1)} .$$

Since for S^6 we obviously have that $c_1 c_2 = c_1^3 = 0$ and $c_3 = 2$, it implies $s_{(3,0,0)} = 2$, $s_{(1,1,0)} = -s_{(0,0,1)} = 6$ what gives formula (49).

8.4.1. *On stable complex structures.* If now c_τ is an arbitrary stable complex structure on S^6 , equivariant under the given action of T^2 , than Corollary 7 implies that the weights at the fixed points of this action related to c_τ are given with

$$a_1(1)\alpha_1, a_2(1)\alpha_2, a_3(1)\alpha_3, \text{ and } a_1(2)(-\alpha_1), a_2(2)(-\alpha_3), a_3(1)(-\alpha_2) .$$

By Corollary 8 the signs at fixed points are

$$\text{sign}(i) = \epsilon \cdot \prod_{j=1}^3 a_j(i), \quad i = 1, 2 .$$

Corollary 9 implies that the coefficients $a_j(i)$ should satisfy the following equations:

$$a_1(1) + a_1(2) = a_2(1) + a_2(2) = a_3(1) + a_3(2), \quad a_1(1)a_2(1) - a_1(2)(a_1(1) - a_2(1)) = 1 .$$

These equations have ten solutions which we write as the couples of triples that correspond to the fixed points. Two of them are $(1, 1, 1)$, $(1, 1, 1)$ and $(-1, -1, -1)$, $(-1, -1, -1)$ and correspond to an invariant almost complex structure J and it's conjugate. The other eight couples are of the form (i, j, k) , $(-i, -j, -k)$ where $i, j, k = \pm 1$ and they describe the weighs of any other stable complex structure on S^6 equivariant under the given torus action. Note that, since $\tau(S^6)$ is trivial bundle, we have on S^6 many stable complex structures different from J . The fact that, for any T^2 -equivariant stable complex structure on S^6 different from J or it's conjugate, the weights at two fixed points differ by sign, together with Proposition 1 proves the following Proposition.

Proposition 7. *The cobordism class for S^6 related to any T^2 -equivariant stable complex structure that is not equivalent to G_2 -invariant almost complex structure is trivial. It particular, besides of described G_2 -invariant almost complex structure, S^6 does not admit any other almost complex structure invariant under the canonical action of T^2 .*

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